RENORMALIZATION GROUP
and
FUNCTIONAL SELF-SIMILARITY

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Contents

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Part 1: General

Chapter I

INTRODUCTION into RENORMALIZATION GROUP

1 History of ideas

The renormalization group (RG) had been first discovered about forty five years ago by Stückelberg and Petermann [1] in the "depth" of quantum field theory (QFT) by studying the structure of an arbitrariness in a finite results of QFT calculations, the results obtained in the course of a complicated procedure of renormalization, that is 'removing of ultraviolet infinities'. Quite soon, it was adequately formulated [2] and successively used [3, 4, 5] in QFT by Nicolai Bogoliubov and the present author for improving approximate perturbation solutions to obtain a correct structure of ultaviolet and infrared asymptotic behaviour, i.e., of solution singularities. Here, like in Ref. [1], the formalism and terminology of continuous groups (of Sophus Lie theory) was used. In this, the Bogoliubov – Stückelberg formulation, the underlying symmetry is an *exact* property of QFT solution[1].

In the seventies, it was found that the RG concept is fruitful also in some other fields of microscopic physics: phase transitions in large statistical systems, polymers, turbulence, and so on. However, here, following the Gell-Mann – Low [6] and the Wilson approach to a spin lattice[7], the original exact symmetry underlying the renormalization group notion in QFT was substituted by an *approximate* one with the related transformations forming a semi-group (not an exact group as in the QFT case). To distinguish between them, in what follows, by RG-symmetry or *renormgroup* symmetry we denote the original exact property of a solution – as it was distinctly formulated in Refs.[2, 5] (see also [8]) by Bogoliubov and the present author for the QFT on the base of finite Dyson transformations.

Later on, the symmetry similiar to the renormgroup one was also found in a number of problems of macroscopic physics like, e.g., mechanics, transfer theory, hydrodynamics and a close relation of renormgroup symmetry to the notion of self-similarity has been

[1] In the popular Gell-Mann – Low approach [6] the corresponding symmetry was treated as an approximate one related only to short-distance, that is to UV, asymptotic behaviour in QFT.

established [9, 10].

2 Mathematical preliminaries

2.1 Renorm-group Folklore in QFT

Let us start with some simple statements which can be supposed to be widely known among the particle theory practitioners. In QFT, the RG usually is associated with possibility to present some physical quantity (e.g., matrix element), $F(Q^2, g)$, calculated under a definite renormalization prescription in a form $F(Q^2/\mu^2, g_\mu)$ (for simplicity we take the massless case) with renormalized coupling constant g_μ definition attached to some renormalization point (or renormalization momentum scale) $Q = \mu$. Differential RG equation is usually said to be driven from the condition that F does not depend on the choice of μ ,

$$\frac{dF}{d\mu} = 0 \ . \tag{1.1}$$

The coupling constant g_μ dependence on μ, normalization momentum value, is described by a specific function $\overline{g}(Q^2)$ known as an *effective coupling* (sometimes – effective coupling constant):

$$g_\mu = \overline{g}(\mu^2) \ .$$

Equation (1) can be written down in the form of partial linear differential equation (DEq)

$$\left[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} \right] \ F(x, g) = 0 \quad . \tag{1.2}$$

where $x = Q^2/\mu^2$, g stands for g_μ and $\beta(g)$, the group generator, usually referred to as *beta function*, is defined by

$$\beta(g_\mu) = z\frac{\partial \overline{g}(z)}{\partial z} \qquad \text{at} \qquad z = \mu^2 \ . \tag{1.3}$$

It is convenient to represent an effective coupling (EC) $\overline{g}$ as a function of two arguments: $Q^2/\mu^2 = x$ and g_μ) with the property $\overline{g}(1, g) = g$. Besides eq.(1.2), $\overline{g}$ satisfies the nonlinear DEq

$$x\frac{\partial \overline{g}(x, g)}{\partial x} = \beta(\overline{g}(x, g)) \quad , \tag{1.4}$$

that is nothing else but characteristic equation for (1.2). To employ this formalism one has to give the $\beta(g)$. Usually for this one uses renormalized perturbation theory.

On the other hand, for some other QFT nonobservable objects $s(x, g)$ depending on only one invariant mommentum squared argument, like scalar propagator amplitude or structure function moment, which are not RG invariant the differential equation is usually written down in the form

$$\left[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} + \gamma(g) \right] \ s(x, g) = 0 \quad . \tag{1.5}$$

with $\gamma(g)$ known as anomalous dimension of $s(x,g)$.

The foregoing can be considered as a "RG folklore". We gave it in the simplest *massless* version, that corresponds to the UV case, for the QFT model with one coupling constant.

2.2 *Group Functional Equation*

Less popular are functional equations (FEs). Such equation for the $\overline{g}$ in the UV case has the form

$$\overline{g}(x,g) = \overline{g}\left(\frac{x}{t}, \overline{g}(t,g)\right) . \tag{1.6}$$

This equation which follows[2] from Dyson renormalization transformations represents a basement of RG formalism. Popular DEq (1.4) can be directly obtained from it by differentiating over x and putting then $t = x$. On the other hand by differentiating (1.6) with respect to t at $t = 1$ we get partial DEq

$$[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g}]\ \overline{g}(x,g) = 0 . \tag{1.7}$$

analogous to eq.(1.2).

Hence the FEq (1.6), as well as similar FEqs for dressed propagators, like, e.g.,

$$\bar{s}(x,g) = \bar{s}(t,g)\bar{s}\left(\frac{x}{t}, \overline{g}(t,g)\right) . \tag{1.8}$$

and vertex functions must be considered as the most adequate and general formulation of RG symmetry in QFT.

However, in reality group FEs, like (1.6), do not content any physics at all being just the reflection of nothing else but the group composition law! Here we mean the group of transformations based upon operation of change a reference point μ involved into definition of a coupling constant g_μ.

Namely, we can regard the change of a reference coupling

$$g_\mu \to g_{\bar{\mu}}$$

as an operation of group element T_t depending on a real continuous positive numerical parameter $t(=\mu/\bar{\mu})$ defined as

$$T_t g_\mu = g_{\mu t} = \overline{g}(t, g_\mu) .$$

If we set $x = \tau t$, then the l.h.s. of (1.6) can be obtained from g by $T_{\tau t}$, while the r.h.s. may be identified with $T_\tau T_t g$. The content of Eq. (1.6) now is just the group composition law,

$$T_{\tau t} = T_\tau T_t .$$

Thus the essence of the basic RG equation (1.6) is the formal condition for transformations T_t to form a group.

[2] See, e.g., Chapter *Renormalization Group* in Ref.[8] .

2.3 *Abstract Formulation*

To make this point more clear, let us show that RG equation (1.6) can be formally obtained directly from the group law. Generally the mathematical formulation of RG transformation can be presented as an explicit functional realization of the mentioned Lie group.

Consider the transformation $T(l)$ of a certain abstract set $\mathcal{M}$ of elements M_i into itself depending on a continuous real parameter l $(-\infty < l < \infty)$ such that for each element M we have

$$T(l)M = M' \ (M, M' \subset \mathcal{M}) \ .$$

Suppose that $\mathcal{M}$ can be projected onto the real axis, i.e. , for every M_i there corresponds some real number g_i. [3] Then this transformation can be written in the explicit analytic form

$$T(l)g = g' = G(l, g) \ ,$$

G being a continuous function of two arguments satisfying the normalization condition

$$G(0, g) = g$$

which corresponds to the identity transformation $T(0) = \mathbf{E}$.

Transformations $T(l)$ form a group if they satisfy the composition law

$$T(l) \times T(\lambda) = T(l + \lambda)$$

to which there corresponds the functional equation for G :

$$G\{\lambda, G(l, g)\} = G(\lambda + l, g) \ . \tag{1.9}$$

As it follows from basic theorems of the Lie group theory, in this equation it is sufficient to deal with the infinitesimal transformation when $\lambda \ll 1$, i.e., with the DEq

$$\frac{\partial G(l, g)}{\partial l} = \beta\{G(l, g)\} \ . \tag{1.10}$$

Here the group generator is defined as

$$\beta(g) = \frac{\partial G(\epsilon, g)}{\partial \epsilon} \quad \text{at} \quad \epsilon = 0.$$

Performing a logarithmic change of variables

$$l = \ln x, \quad \lambda = \ln t \ , \quad G(l, g) = \bar{g}(x, g) \ , \quad T(\ln t) = T_t$$

we obtain Eqs. (1.6),(1.4) instead of (1.9) and (1.10).

[3] This condition is not essential and can be modified - see below Eqs (1.17)-(1.19).

3 Definition of the Renorm-Group

3.1 *Introductione*

In the simplest case the renormalization group can be defined as a continuous one-parameter group of specific transformations of some solution of a physical problem, solution that is fixed by a boundary condition. The RG transformation involves boundary condition parameters and corresponds to some change in the way of imposing this condition.

For illustration imagine some one-argument characteristic $f(x)$ of a solution, characteristic that can be fixed by the boundary condition $f(x_0) = f_0$. Represent formally this characteristic as a function of boundary parameters as well: $f(x) = f(x, x_0, f_0)$. (This step can be considered as an *embedding* operation). The RG transformation then corresponds to a changeover of the way of parameterization, say from $\{x_0, f_0\}$ to $\{x_1, f_1\}$. In other words the x argument value at which the boundary condition is given does not need to be x_0, but we may choose another point x_i. Suppose now that our characteristic f can be written in a form of two-argument function $F(x/x_0, f_0)$ with a property $F(1, \gamma) = \gamma$. The following equality

$$F(\frac{x}{x_0}, f_0) = F(\frac{x}{x_1}, f_1) \quad .$$

depends on the fact that under such change of a boundary condition the form of the function F itself is not modified (as, e.g., in the case of $F(x, \gamma) = \Phi(\ln x + \gamma))$. Noting $f_1 = F(x_1/x_0, f_0)$ and in terms of $\xi = x/x_0$ and $t = x_1/x_0$ we obtain the functional equation

$$F(\xi, f_0) = F(\xi/t, F(t, f_0)) \ .$$

The group operation now can be defined as

$$G_t \ : \ \{ \ \xi \to \xi/t \ , \quad f_0 \to f_1 = F(t, f_0) \ \} \ .$$

3.2 *RG-transformation*

The renorm-group transformation for a given solution of some physical problem in the simplest case can be defined as a one-parameter transformation of its two characteristics, say q and g, by

$$R(l) \quad : \quad \{ \ q \to q' = q - l \ , \quad g \to g' = G(l, g) \ \} \ , \tag{1.11}$$

the first being a translation and the second of a more complicated functional form. Eq. (1.9) for the transformation function G provides the group property of the whole transformation (1.11).

Performing the logarithmic change of variables

$$q \to x = e^q, \quad l \to t = e^l$$

and the appropriate redefinition of a transformation function

$$G \to \bar{g}(x,g) = G(\ln x, g)$$

we obtain Eqs.(1.4) and (1.6)and transformation

$$R_t \quad : \quad \{ \, x' = x/t \ , \quad g' = \bar{g}(t,g) \, \} \tag{1.12}$$

instead of eqs.(1.9), (1.10) and (1.11). One can refer to these eqs. (1.4), (1.6) and (1.12) as to the *multiplicative* version (and previous Eqs. as to an *additive* one). They are just RG transformation and equations for the effective coupling $\bar{g}$ in massless QFT with one coupling g. In that case $x = Q^2/\mu^2$ is the ratio of a 4-momentum Q squared to a "normalization" momentum μ squared and g, the coupling constant.

Usually of interest are also some functions $\phi(q,g)$ transforming as a linear representation of RG :

$$\phi(q,g) \ \to \ R(l)\phi = z(l,g)\phi(q',g') \ . \tag{1.13}$$

Several simple generalizations of (1.12) will be considered below.

3.3 *Simple Generalizations*

"Massive" Case. For example in QFT, if we do not neglect the particle mass m, we should insert an additional dimensionless argument into the effective coupling $\bar{g}$ which now has to be considered as a function of three variables $x = Q^2/\mu^2, \ y = m^2/\mu^2$, and g. The presence of a new "mass" argument y modifies the group transformation

$$R_t \ : \ \{ \, x' = x/t \ , \quad y' = y/t \ , \quad g' = \bar{g}(t,y;g) \, \} \tag{1.14}$$

and the functional equation

$$\bar{g}(x,y;g) = \bar{g}\left(\frac{x}{t} \ , \ \frac{y}{t} \ ; \ \bar{g}(t,y;g) \right) \ . \tag{1.15}$$

Here, it is important that the new parameter y (that in physical nature must be close to the x variable as it scales in the same way) enters also into the transformation law of g .

If in the considered QFT model there are several masses (like, e.g., in QCD) there will be several mass arguments

$$y \to \{y\} = y_1, y_2, \ldots y_n \ . \tag{1.16}$$

In what follows we shall use eqs.(1.14), (1.15) keeping the possible change (1.16) in mind.

Multi-coupling case. Another generalization corresponds to the transition to the several coupling constants case:

$$g \to \{g\} = g_1, \ldots g_k \ .$$

Here one has to introduce the "family" of effective couplings

$$\bar{g} \to \{\bar{g}\} \ , \quad \bar{g}_i = \bar{g}_i(x, y; \{g\}) \ , \qquad i = 1, 2, \ldots k \ , \tag{1.17}$$

satisfying the system of coupled functional equations

$$\bar{g}_i(x, y; \{g\}) = \bar{g}_i \left(x/t \ , \ y/t \ ; \ \ldots \ \bar{g}_j(t, y; \{g\}) \ \ldots \ \right) \ . \tag{1.18}$$

This system is a generalization of (1.6) and (1.15) to the case when every element M_i of $\mathcal{M}$ can be described by k numerical parameters, i.e., by the point $\{g\}$ in a k-dimensional real parameter space. The RG transformation for this case looks like

$$R_t \ : \ \{ \ x \to x/t \ , \quad y \to y/t \ , \quad \{g\} \to \{\overline{g}(t)\} \ \} \ , \tag{1.19}$$

with

$$\overline{g}_i(t) = \overline{g}_i(t, y; \{g\}) \ .$$

4 First Illustration: Elastic Rod

As can be shown, the symmetry underlying the FSS group transformations can be 'discovered' in many theoretical problems taken from diverse fields of physics. Consider a very simple example from statics.

Figure I.1: "Elastic rod" model

Imagine an elastic rod with a fixed point (the point "0" in Fig. I.1) bent by some external force, e.g., gravity or pressure of a moving gas or liquid. The form of the rod can be described by the angle g between the tangent to the rod and the vertical direction considered as a function of distance l along the rod from a fixed point, that is by the function $g(l)$. If the properties of the rod material as well as external forces are homogeneous along its length (i.e. independent of l), then $g(l)$ can be expressed as a function $G(l, g_0)$ depending also on g_0, the deviation angle at the point held fixed from which the distance l is measured. Naturally, G should depend on some other arguments, like extra forces and rod material parameters, as well but in this context they are irrelevant.

Consider two arbitrary points on the rod, "1" and "2" (see Fig. I.1 with coordinates $l_1 = \lambda$ and $l_2 = \lambda + l$. The angles g_i at points "0", "1" and "2" can be related via G function :

$$g_1 = G(\lambda, g_0), \quad g_2 = G(\lambda + l, g_0) = G(l, g_1) \ . \tag{1.20}$$

Note here that to get the last form of r.h.s. of the second equation, we have to imagine that the point held fixed is "1" instead of "0" as in Fig. I.2. Combining all these Eqs.

Figure I.2: Group operation for the "Rod".

we come to the group composition law (1.9) for the function $G(l, g)$.

In course of deriving the second of Eqs. (1.20) we have tacitly assumed that the rod is of the infinite length. If we introduce a finite length L - see Fig. I.3, then the $G(l, g)$

Figure I.3: Rod with discrete inhomogeneity.

dependence must be replaced by function $G(l, L, g)$ of three essential arguments, where the second one is the distance between the fixed point and the free end.

Having now

$$g_1 = G(\lambda, L, g_0), \quad g_2 = G(\lambda + l, L, g_0) = G(l, L - \lambda, g_1) \tag{1.21}$$

we come to the functional equation

$$G(l+\lambda, L, g) = G(l, L-\lambda, G(\lambda, L, g)) \tag{1.22}$$

which is just the "additive" version of (1.15) and can be transformed to it by an appropriate change of variables used above to get (1.9) from (1.6).

Let us note that in (1.21) and (1.22) the second argument is not necessarily the rod length. It can be considered as a distance from a fixed point to the place in which the rod properties undergo a discrete change (say, in thickness or material). Generally, the additional argument L describes the *discrete breaking of homogeneity* property of the considered system. Such a breaking can take place at several points. Their coordinates must be introduced as G additional arguments: $L \to \{L\}$. In the QFT case this corresponds to the introduction of particle masses.

Chapter II

RENORMGROUP and FUNCTIONAL SELF-SIMILARITY

1 Functional self-similarity

1.1 *Self-similarity*

The RG transformations discussed above have close connection with the concept of self-similarity (SS) in mathematical physics. The self-similarity transformation for a problem formulated by nonlinear partial DEs are well known since last century mainly in hydrodynamics of liquids and gases. It is a simultaneous one-parameter λ transformation defined as power scaling of variables $z = \{x, t, \ldots\}$ and functions $V_i(x, t, \ldots)$, e.t.c. :

$$S_\lambda : \{x \to x\lambda \quad , t \to t\lambda^a \quad , \; V_i(z) \to V_i'(z') = \lambda^{\nu_i} V_i(z') \quad \} \quad .$$

To emphasize their power structure we shall use a term *p̲ower s̲elf-s̲imilarity* =PSS. According to Zel'dovich and Barenblatt [11] the PSS can be divided in two classes:

a/ PSS of the 1st kind

with all exponents $a, \nu, ...$ being integers or rational ("Integer PSS"). Usually, these could be determined from the theory of dimensions;

b/ PSS of the 2nd kind

with irrational exponents ("Fractal PSS") which should be defined from dynamics.

1.2 *Renormgroup vs self-similarity*

To relate RG with PSS let us turn to the solution of the basic renorm-group FEq

$$\overline{g}(xt, g) = \overline{g}(x, \overline{g}(t, g)) \quad . \qquad (1.5')$$

Its general solution is known; it depends on arbitrary function of one argument – see below Eq.(3.12). However, at the moment we are interested in a special solution linear in the second argument g

$$\overline{g}(x,g) = gf(x) \ .$$

Function $f(x)$ should satisfy equation $f(xt) = f(x)f(t)$, whose solution is $f(x) = x^{\nu}$ and

$$\overline{g}(x,t) = gx^{\nu} \ .$$

This means that in our special case the RG transformation (1.12) is reduced to PSS transformation,

$$R_t : \{x \to xt^{-1}, \ g \to gt^{\nu}\} = S_t \quad . \tag{2.1}$$

Generally, in RG, instead of a power law, we have arbitrary fuctional dependence. Hence, one can consider transformations (1.12), (1.15),(1.19) as *functional* generalizations of usual (i. e. power) self-similarity transformations. It is natural, in general context, to refer to them as to the transformations of functional scaling or functional self-similarity (FSS) rather than to RG-transformations. In short

$$\mathrm{RG} \equiv \mathrm{FSS} \quad ,$$

where FSS stands for "Functional Self-Similarity".

2 Other illustrations from classical Physics

2.1 *Weak shock wave*

Another example can be taken from hydrodynamics. Consider a weak shock wave in the one-dimensional case. The velocity-distance plot (distance from the creation point) at the given moment has a triangular shape as in Fig. II.1

Figure II.1: Weak shock wave shape.

and can be described by the expression

$$v(l) = \frac{l}{L}V \ , \quad l \le x, \quad = 0 \ l > \ x. \tag{2.2}$$

where $x = x(t)$ is the front position and $V = v(x)$ – the front velocity. In the absence of viscosity the "consevation law" $LV =$ const. holds. Here L and V are the coordinate and the velocity at the shock wave front. They are functions of time. However they can be treated as functions of the front wave position $L = x, V = V(x)$ as well. If the physical situation is homogeneous then the front velocity $V(x)$ should be considered as a function of two additional relevant arguments – its own value $V_0 = V(x_0)$ at some precedent point $(x_0 < x)$ and of x_0 coordinate. In can be written down in the form : $V(x) = G(x/x_0, V_0)$. If we pick up three points x_0, x_1 and x_2 (as in the Fig. II.2 under homogeneity surrounding the initial condition may be given either at x_0 or x_1. Thus we

Figure II.2: Weak shock wave evolution.

obtain the FSS equation analogous to (1.20)

$$V_2 = G(x_2/x_0, V_0) = G(x_2/x_1, V_1) = G(x_2/x_1, G(x_1/x_0, V_0)) \ .$$

2.2 *Transfer theory*

A similar argument has been done by Mnatzakanian [13] in the one-dimensional transfer problem. Consider a half-space filled with a homogeneous medium, on the surface of which some flow (of radiation or particles) with the intensity g_0 falls from the vacuum (see Fig. II.3.

Figure II.3: One – dimensional transfer.

Let us follow the flow as it moves inside the medium at the distance l from the boundary. Due to homogeneity along the l coordinate (the distance from boundary), the intensity of a penetrated flow $g(l)$ can be represented as a function of two essential arguments, $g(l) = G(l, g_0)$.The values of the flow at points "O", "1" and "2" shown in Fig. II.3 can be connected with each other by the transitivity relations,

$$g_1 = G(\lambda, g_0) \quad , \quad g_2 = G(\lambda + l, g_0) = G(l, g_1) \quad , \tag{1.20}$$

which lead to Eq. (7), i.e., the additive version of FSS FE.

This problem admits a generalization connected with a discrete inhomogeneity, that is with the case of two different kinds of homogeneous materials separated by the inner boundary surface at $l = L$ as in Fig. II.4. As in the elastic rod case, the point of

Figure II.4: Transfer with inner boundary.

homogeneity breaking $l = L$ may correspond to the boundary with empty space, and resulting equation coinsides with eq. (1.22).

The transfer problem admits another generalization described by the "multiplication" of the last argument as expressed by eq.(1.17). Consider the case of radiation of two different frequencies ω_1 and ω_2 (or particles of different energies or of different types) as shown in Fig. II.5.

Figure II.5: "Two - component" transfer.

Suppose that the material of the medium has such properties that the transfer processes of the two flows are not independent. In that case, the characteristic functions of these flows G and H are dependent on both boundary values g_o, h_o and can be considered as three-argument functions

$$g(l) = G(l, g_0, h_0) \quad , \quad h(l) = H(l, g_0, h_0) \quad .$$

After group operation $l \to l - \lambda$ we arrive at coupled set of functional equations:

$$G(l+\lambda, g, h) = G(l, g_\lambda, h_\lambda) \quad , H(l+\lambda, g, h) = H(l, g_\lambda, h_\lambda) \quad ,$$

$$g_\lambda \equiv G(\lambda, g, h) \quad , \quad h_\lambda \equiv H(\lambda, g, h) \quad .$$

which is nothing else but an additive version of system Eq. (1.18) for $k = 2$.

Now we can make the important conclusion that a common property yielding functional group equations is just the transitivity property of some physical quantity with respect to the way of giving its boundary or initial value.

Hence, the RG or FSS symmetry is not a symmetry of equations but a symmetry of solution (or of equations and boundary conditions considered as a whole).

3 RG in QED

3.1 *Effective Electron Charge*

An essential feature of QFT is the presence of virtual states and of virtual transitions. In QED, for example, the process of virtual dissociation of a photon into an electron-positron pair and vice versa $\gamma \leftrightarrow e^+ + e^-$ can take place. The sequence of two such consecutive transitions can be drawn as a simple QED Feynman diagram presented in Fig. II.6.

Figure II.6: Vacuum polarization in QED.

The vacuum polarization processes lead to several specific phenomena and particularly to the notion of *effective electron charge*. To explain this, let us start with a classical analogy.

Consider a polarizable medium consisting of molecules that can be imagined as electric dipoles. Let us insert into it an external electric charge Q. Due to the attraction of opposite charges, the elementary dipoles change their position so that this charge Q turns out to be partially screened. As a result, at a distance r from Q the electric potential will be smaller than the vacuum Coulomb value Q/r and can be presented in the form $Q(r)/r$ where ganarally $Q(r) \leq Q$. The introduced quantity $Q(r)$ is known as an effective charge. As r decreases, $Q(r)$ increases and as $r \to 0, Q(r)$ tends to Q.

In QFT, the vacuum, i.e., the interparticle space itself, stands for the "polarizable medium". Quantum-field vacuum, physically, is not empty. It is filled with vacuum fluctuations, i.e., with virtual particles. These "zero fluctuation" are a well-known property of a ground state in quantum world. In QED, zero oscillations consist mainly of short-lived virtual (e^+, e^-) pairs that play the role of tiny electric dipoles. The influence of fluctuations of that sort will be considered here in detail.

Figure II.7 shows Feynman diagrams that describe schematically the process of measuring the electron charge. The electron under measurement is denoted by the symbol E. The measurement is performed with the help of some external electromagnetic field A_{ext}.

Figure II.7: Electron charge measurement: a) classic, b) quantum .

Figure II.7a represents just the classical picture without quantum effects. In Fig. II.7b, the case when the probing photon (that is the quantum of external field) virtually dissociates into (e^+, e^-) pair is presented. This pair forms a virtual dipole that yields partial screening of the measured charge. As far as this process involves two elementary electromagnetic interactions, its contribution into effective charge is proportional to the small number $e^2 \equiv \alpha \simeq 1/137$; and this contribution depends on the distance r ! In the region of r values much smaller than the Compton length of the electron $r_e = h/mc \simeq 3,9.10^{-11} cm$, it depends on distance r logarithmically and is described by the expression

$$e \to e(r) = e\{1 - \frac{\alpha}{3\pi} \ln\frac{r}{r_e} + O(\alpha^2)\}.$$

Here the terms $\sim \alpha^2$ are related to processes presented on Figs. II.7c,d, and so on.

It is important to note that the phenomenon of the electron charge dependence on the distance has a purely quantum origin (as it was first discussed by Dirac [14] in the middle of the 30-ies). The $e(r)$ value decreases as r grows. So, quantitatively, the QED effective-charge behavior corresponds to a classical picture of screening. This dependence is presented in Fig. II.8 by a set of curves. Each curve corresponds to a possible behavior of the effective charge $e(r)$ as obtained from the theory and considered without any reference to experiment (parameter e, $\alpha = e^2$ being unspecified numerically).

Figure II.8: Theoretical possibilities for electron effective charge.

The point is that in the classical analog the value of an external charge Q, that is inserted into the polarizable medium, is known "from the very beginning" (from the independent measurement). In our quantum case it is not so and charge value can be measured only at *not very small* distances. The result of measurement, generally, can be specified by two quantities: the "distance of a measurement" r_i and the measured charge value e_i. Hence, to make the choice from the set of curves of Fig.(II.8) one has to fix a point on the plane with coordinates $r = r_i$, $e(r) = e_i$. Thus, for this chosen curve $e(r_i) = e_i$. Note, that the usual definition of an electron charge by macroscopic type (like Millikan) experiments corresponds here to very large distances $r \gg r_e$, i.e., $1/137 = e^2(r = \infty)$.

As it is well known, in microphysics one usually exploits the momentum, rather than coordinate, representation. Correspondingly, instead of $e(r)$ one deals with the quantity $\bar{\alpha}(Q^2)$, i.e., with a Fourier transform of $e(r)$ squared. It is a monotonically increasing function of its argument Q^2, that is, of the 4-momentum transfer squared. Here and below, the bar denotes a function (distinct from α, α_μ, α_i - its numerical values at some given value of the Q^2 argument). The correspondence condition with the classical electrodynamics now takes the form $\bar{\alpha}(0) = 1/137$, as in our scale the classical field corresponds to a photon with zero 4-momentum. However, as before, to fix one of possible curves on the plane $(Q, \bar{\alpha})$ one has to give the point $Q = \mu$, $\bar{\alpha} = \alpha_\mu$ and, hence, for the selected curve (that is presented in Fig (II.9) one has $\bar{\alpha}(\mu^2) = \alpha_\mu$. The parameter μ sometimes is referred to as a *scale parameter*. As is clear, it is just

Figure II.9: Various possibilities for parameterization ... .

the momentum of the photon magnitude used for a charge measurement. The effective coupling function $\bar{\alpha}(Q^2)$ describes the dependence of the electron charge value on the measurement conditions. It is worth noting that in our days the logarithmic corrections to the Millikan value become essential and can be measured at big accelerators.

The parameter μ has no analogue in classical electrodynamical Lagrangian. The phenomenon of its arising in QFT is connected with the term "dimensional transmutation". As it is shown above, its appearance is very natural. This is a good place to recall the well-known ideas by Niels Bohr formulated in the middle of thirties [15] and related to the complementarity principle. The point is that to specify the quantum system, it is necessary to fix its "macroscopical surrounding", i.e., to give the properties of macroscopic devices used in the measurement process. Just these devices are described by additional parameters. However, this is not the end of the Bohr (i.e., – scale) parameter story. As can be shown, its existence leads to a new symmetry lying in the foundation of the renormalisation group.

3.2 *RG transformations*

To do this, consider again Fig. II.9 having in mind that a choice of one curve has been made by the condition $\bar{\alpha}(Q^2 = \mu^2) = \alpha_\mu$. Assume also for simplicity that we deal with a massless QED, more precisely, with the approximation $\mid Q \mid \;\gg mc$. This corresponds to the GeV-energy region or to distances $r \;\ll\; r_e$. Here, the dimensionless effective charge function $\bar{\alpha}$ can be represented as a function of two dimensionless arguments Q^2/μ^2 and α_μ, i.e.,

$$\bar{\alpha}(Q^2) \;=\; \bar{\alpha}(Q^2/\mu^2, \alpha_\mu)$$

. Now let us take into account that the pair of parameters μ, α_μ used for identifying the physical curve may, generally, correspond to any point on the curve. Take two points "1" and "2" on this curve with coordinates μ_1, α_1 and μ_2, α_2, respectively. It is evident that $\bar{\alpha}$ can be parameterized by any pair $\mu_i, \alpha_i;\;\; i = 1, 2, \ldots$, so that for arbitrary Q^2 value the identity

$$\bar{\alpha}(Q^2/\mu_1^2, \alpha_1) = \bar{\alpha}(Q^2/\mu_2^2, \alpha_2)$$

should hold.

At the same time the second argument in the r.h.s., α_2 which by definition is equal to $\bar{\alpha}$ at $Q^2 = \mu_2^2$, can be expressed in terms of $\bar{\alpha}$ parameterized with the help of point "1" coordinates, i.e.

$$\alpha_2 = \bar{\alpha}(Q^2 = \mu_2^2) = \bar{\alpha}(\mu_2^2/\mu_1^2, \alpha_1).$$

Combining the last two equations and introducing a simplified notations

$$Q^2/\mu^2 = x, \qquad \alpha_1 = \alpha, \qquad \mu_2^2/\mu_1^2 = t$$

we arrive at the functional equation

$$\overline{\alpha}(x, \alpha) = \bar{\alpha}(x/t, \bar{\alpha}(t, \alpha)), \tag{2.3}$$

identical with Eq. (5) lying, as was mentioned above in Section 1.2 , in the foundations of RG mathematical formalism.

Note that the corresponding continuous one-parameter transformation is just the change ("1" → "2") of the parameterization point

$$R_t : \{\mu_1 \to \mu_2 = \sqrt{t}\mu_1, \ \alpha_1 \to \alpha_2 = \overline{\alpha}(t, \alpha_1)\} \quad . \tag{2.4}$$

as explicitly indicated in the Fig. II.10.

Figure II.10: Change of parameterization ... .

Thus we have shown that in the renormalised QED there exists an invariance with respect to continuous transformations of the group type which involve two quantities and contain functional dependence. Going to the dimensionless arguments we can write them down in the form (I-10b).

As can be shown, in QED the effective coupling $\bar{\alpha}$ is equal to product of α and dimensionless function $d(x, \alpha)$ describing the transverse photon propagator with due regard for vacuum polarization effects. Generally, in QFT models with one coupling constant the effective coupling $\bar{g}(x, g)$ can be expressed as a product of g, corresponding vertex function and propagators of different fields . Usually this can be done on the basis of Dyson renormalization transformations.

3.3 *Dyson Transformation*

As it is well known from the QFT renormalization theory, transition from one renormalization scheme (RS) to another can be formulated in terms of multiplicative Dyson transformations for vertices V_i and propagators D_k:

$$V_i \to V_i' = z_i^{-1} V_i, \quad D_k \to D_k' = z_k D_k$$

Here one must treat V and D as renormalised ones and all z as finite real numbers.

Dyson transformation is generally equivalent to an effective change of operator field functions normalization and at the very end to that of coupling constants. As for a proper definition of these constants, one must specify the kinematic condition of their possible measurements. This means that vertices and propagators corresponding to different renormalization prescriptions can be related by a suitable change of momentum arguments.

With due regards for this momentum rescaling, it is possible to rewrite the Dyson transformations in the momentum representation as follows (see e.g. Section 47 in the [8] or Appendix IX of [16]

$$s_i(\frac{k^2}{\lambda_2^2}, \frac{m^2}{\lambda_2^2}, g_2) = z_i s_i(k^2/\lambda_1^2 a, m^2/\lambda_1^2, g_1),$$

$$V_j(\{k^2/\lambda_2^2\}, \frac{m^2}{\lambda_2^2}, g_2) = z_j^{-1} V_j(\{k^2/\lambda_1^2\}, m^2/\lambda_1^2, g_1),$$

with

$$g_2 = z_v \Pi_i \{z_i^{-1/2}\} g_1$$

and $\{k^2\} = k_1^2, k_2^2, \dots k_n^2$. Taking into account normalization conditions for scalar functions s_i, V_j, which in the RS based upon the momentum subtraction (MOM-scheme) can be chosen in a very simple form

$$s_i(1, \frac{m^2}{\lambda^2}, g) = V_j\left(\{1\}, \frac{m^2}{\lambda^2}, g\right) = 1$$

one can define finite renormalisation constants z_i, z_j in terms of s_i and V_j. Using expressions thus obtained it is easy to get functional equations of the type [compare with Eq.(1.13)]

$$s_i(x, y; g) = s_i(t, y; g) s_i(x/t, y/t; \bar{g}(t, y; g)) \ , \tag{2.5}$$

where $\bar{g}$, the so called *invariant coupling*, satisfies a more simple closed equation

$$\bar{g}(x, y; g) = \bar{g}(x/t, y/t; \bar{g}(t, y; g)) \ . \tag{I.13}$$

Note here that the normalisation conditions

$$s_i(1, y, g) = 1 \quad \text{and} \quad \bar{g}(1, y, g) = g \tag{2.6}$$

are "built in" functional equations (2.5) and (1.15).

Note also that group FEs for matrix element can be written as follows

$$M(\{x\}, y; g) = M(\{x/t\}, y/t, g(t, y; g)) \tag{2.7}$$

that reflects its independence of remormalization details.

Thus the RG invariance is nothing but the invariance with respect to the way of parameterization. For instance, instead of using the "Millikan's value" $\overline{\alpha}(0) = 1/137$ one may take a more nice number

$$\alpha_{cern} = \overline{\alpha}(M_Z^2) = \frac{1}{128} = 2^{-7} \ ,$$

– the "CERN value" at $Q = M_Z$ as well – see Fig. II.11.

Figure II.11: Various parameterizations of QED coupling.

3.4 *The Nature of RG=FSS Symmetry*

Now we can try to answer the question about the nature of a symmetry underlying the functional self-similarity and renorm-group transformations.

What is the meaning of $RG \equiv FSS$ *symmetry?*

As can be easily seen from the examples presented in the Sections 1d and 2a it is **not** a symmetry of equation(s) of a given physical problem but rather

> the symmetry of solution(s) considered as a function of essential physical variables and of appropriate boundary value(s).

It can be defined as an invariance property of some physical quantity expressed by such solution with the respect to the way of giving its boundary value. The change of this way is just the group operation and generally the group property in this context can be considered as a transitivity property of such operations.

Let us also note that here the important underlying property of a physical system is the homogeneity that can be broken by a discret way. The FSS symmetry is a very simple and usual property in physical phenomena.

Chapter III

RENORMALIZATION GROUP METHOD

1 Formulation of the Method

1.1 *The basic idea*

Approximate solution of the physical problems with FSS = RG symmetry usually does not obey this symmetry which is lost in the course of approximation. This is essential when solution under consideration posseses a singularity as far as the singularity structure usually is destroyed by approximation.

In QFT, e.g.,the usual way of calculation is based on perturbation method, i.e., on power expansion in g. It is not difficult to see that finite sums of this expansion do not satisfy the functional group Eqs. As the simplest illustration consider the effective coupling $\bar{g}$ in the UV region where the one-loop contribution has a logarithmic form

$$\bar{g}_{\mathrm{PTh}}(x,g) = g + g^2\beta\ln x. \tag{3.1}$$

By substituting this expression into FEq. (1.6) after simple manipulation one arrive to

$$Discrp[\bar{g}^{(1)}_{\mathrm{PTh}}] = \bar{g}^{(1)}_{PTh}(x,g) - \bar{g}^{(1)}_{PTh}\left(\frac{x}{t}, \bar{g}^{(1)}_{PTh}(t,g)\right) =$$

$$= [g + g^2\beta\ln x] - [g + g^2\beta\ln x + 2g^3\beta^2\ln t\ln(x/t)] \neq 0$$

- error in the g^3 order. This discrepancy $\sim g^3$ can be liquidated by addition of the next order term $g^3\beta^2\ln^2 x$ into the r.h.s. of (3.1):

$$\bar{g}_{PTh,2} = g + g^2\beta\ln x + g^3\beta^2\ln^2 x.$$

The "improved" expression would yield the discrepancy of the g^4 order which in its turn can be abolished by addition $g^4\ln^3$ term and so on. Thus, we see that, on one hand, finite polynomials cannot satisfy the condition of renormalisation invariance. On the other, we can conclude that functional group equation provides us with a tool for iterative reconstruction of renorm-invariant expression that has a form of infinite series.

This example illustrates a rather general situation. As a rule, approximate solutions do not satisfy a group symmetry. In our case this is happened in the UV limit at $\ln x \to \infty$ where the observed discrepancy becomes important.

Another illustration is provided by the one-dimensional transfer problem discussed in the Section # 2. Here, it is rather simple to get the approximate behavior close to boundary

$$G(l,g) = g + lG'(0,g), \qquad l \ll 1, \tag{3.2}$$

which, being considered for arbitrary l values, also does not obey the FSS symmetry.

On this basis one can put the problem of "renormalization-invariant improvement" of perturbative results. The key idea is to combine such solutions with group equations. The most simple and convenient way for this "marriage" is the use of Lie equations, i.e., group differential equations. The renormalization group method (RGM) is essentially based on this group equations.

1.2 *Differential Formulation*

These equations can be obtained from the functional ones in two different ways. Differentiating Eq. (1.15) by x and putting $t = x$ then, one obtains:

$$x\frac{\partial \bar{g}(x,y,g)}{\partial x} = \beta(\frac{y}{x}, \bar{g}(x,y;g)), \tag{3.3}$$

with

$$\beta(y,g) = \frac{\partial \bar{g}(t,y;g)}{\partial t} \qquad \text{at} \qquad t=1 \tag{3.4}$$

The nonlinear equation (3.3) can be considered as a "massive" generalization of the equation (1.4).

On the other hand, one can differentiate Eq.(1.15) with respect to t at the point $t = 1$, which yields

$$\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g}\right]\bar{g}(x,y;g) = 0 \tag{3.5}$$

- a linear partial differential equation.

Analogous operations applied to Eq.(I.25) lead to:

$$\frac{\partial s(x,y;g)}{\partial\ \mathrm{lnx}} = \gamma[\frac{y}{x}, g(x,y;g)]s(x,y;g) \tag{3.6}$$

and

$$\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g} - \gamma(y,g)\}s(x,y;g) = 0 \tag{3.7}$$

where

$$\gamma(y,g) = \frac{\partial s(t,y;g}{\partial t} \qquad \text{at} \qquad t=1 \tag{3.8}$$

is the so called *anomalous dimension* of s. For a matrix element M satisfying functional Eq.(I.27) this dimension is equal to zero. The corresponding partial differential equation looks like

$$\{\sum_i x_i \frac{\partial}{\partial x_i} - y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g}\}M(x,y;g) = 0 \tag{3.9}$$

Equations (3.5), (3.7), (3.9) express the independence of t variable of the r.h.s. of the corresponding functional group equations, i.e., a mutual compensation of t dependencies via three (or more) arguments. They can be called *compensational* equations to distinguish them from nonlinear Eqs. (3.3), (3.6) which can be referred to as *evolution* group equations.

Stress that compensational as well as evolution differential Eqs. taken together with normalization (i.e. boundary) conditions (2.6) like $\bar{g}(1,g) = g$ are equivalent to functional Eqs. and to each other. At the same time, evolution Lie equations turn out to be more convenient for practical construction of a solution, generators β, γ being given.

It is not difficult to formulate group DEqs for multi-coupling case by proper differentiating FEqs. (1.18). For instance, the system of evolution DEqs looks like

$$x\frac{\partial \bar{g}_i(x,y,\{g\})}{\partial x} = \beta_i(\frac{y}{x}, \{\bar{g}_j(x,y;\{g\})\}), \tag{3.10}$$

Let us comment also that the UV limit of compensational differential Eqs. like e.g.

$$\{x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} - \gamma(g)\}s(x,g) = 0 \tag{II − 7'}$$

coincides with the UV limit of specific nonclosed Eqs. of the type

$$\{x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} - \gamma(g)\}s(x,g) = \Delta S \tag{C − S}$$

obtained in the early 70-ies by Callan and Symansik. The r.h.s. of this equation contains the result of mass counter-term insertion into all internal lines of all diagrams for the function s under consideration. For this reason in current literature compensational Eqs. are usually related to as the Callan-Symansik equations. However, these Eqs. just in the form (3.5), (3.7) were first obtained by Lev Ovsyannikov in 1956 while solving [17] functional RG equations. Therefore we consider it justifable to relate compensational differential Eqs. to the Ovsyannikov's rather than to some other names.

1.3 *General solution*

General solution of the group functional equations was obtained in the mentioned paper[17] by applying the general theory of partial differential equations to the compensational Eqs. (3.5) and (3.7). Details of the derivation can also be found in the Section 48.3 of the monograph[8]. The results obtained can be formulated as follows:

To every solution of the FEq.(3.5) there corresponds some function of two arguments $F(y,g)$, reversible with respect to its second argument and connected to $\bar{g}$ by the relation

$$F(y,g) = F(\frac{y}{x}, \bar{g}(x,y;g)) \qquad (3.11)$$

That is this functional relation represents the general solution of (3.5) as well as (1.15). The explicit form of the $\bar{g}$ can be obtained from eq. (3.11) by reversing the r.h.s:

$$\overline{g}(x,y,g) = F^{-1}_{(2)}[y/x, F(y,g)] \ ,$$

To determine F it is sufficient to specify the generator $\beta(y,g)$.

Note also that to get from the Ovsiannikov result (3.11) the solution in the UV limit, that is in a massless case at $y=0$, one has to assume for F a rather specific limiting form

$$F(y,g) = y\exp[f(g)] \quad \text{or} \quad = \ln y + f(g) \quad \text{as} \quad y = 0 \ .$$

Then

$$f\{\bar{g}(x,g)\} - f(g) = \ln x \ . \qquad (3.12)$$

and

$$\overline{g} = f^{-1}\{\ln x + f(g)\} \ ,$$

Here $f'(g) = 1/\beta(g)$. Solution (3.12) was first obtained by T.D.Lee - see Appendix to the Ref.[6].

To every solution of Eq.(3.7) for an 'one-argument' function s there corresponds some function $\Sigma(y,g)$ related to s by

$$s(x,y;g) = \frac{\Sigma[y/x, \bar{g}(x,y;g)]}{\Sigma(y,g)} \ . \qquad (3.13)$$

Let us give also the general solution of the same type for the system (1.18) at arbitrary finite k value, i.e., for k-couplings case. It can be written down in terms of k arbitrary 3-argument functions F_i , that must be reversible simultaneously with respect to last arguments, and defined from the system of k functional relations

$$F_i(y,\{g\}) = F_i[\frac{y}{x}, ...\bar{g}_i(x,y;\{g\}) \ , \qquad (3.14)$$

with

$$\{g\} = g_1, ...g_k \ \text{ and } \ i,j = 1,...,k \ .$$

Note here that all solutions (3.10) - (3.14) satisfy the normalization condition of the Eq.(2.6) type.

The transition to the massless limit in expressions (3.13) and (3.14) can be performed by a trick analogous to the given above. Then, e.g.,

$$s(x,g) = x^{\sigma}\frac{\Sigma\{\bar{g}(x,g)\}}{\Sigma(g)} \ . \qquad (3.15)$$

Let us also present the solution for the 2-coupling case $g_1 = g, g_2 = h$ in massless limit in the form analogous to (3.12)

$$f_i(\bar{g}, \bar{h}) = f_i(g, h) + \ln x, \qquad i = 1, 2 \ . \tag{3.16}$$

The generalization of this system on a more general case is evident. From the solutions presented it follows that imposing group properties one reduces by unity the number of independent arguments.

1.4 *Technology of RG Method*

The idea of the approximate solution marriage with group symmetry can be realised with help of group DEs. If we define group generators β, γ from these approximate solutions and then solve evolution DEs we obtain *RG improved* solutions that obey the group symmetry and correspond to approximate solutions used as input.

Now we can formulate a method of approximate solution improving. The procedure is given by the following recipe which we illustrate by a massless 1-coupling case (1.4), (1.6):

Assume some approximate solution $\overline{g}_{\rm appr}$ is known.

1. On the basis of the Eq. (1.3) define the *beta*-function with the help of some approximate solution,

$$\beta(g) \stackrel{\rm def}{=} \frac{\partial}{\partial \xi} \overline{g}_{\rm appr}(\xi, g)\Big|_{\xi=1} \ . \tag{3.17}$$

2. Integrate the equation (1.4), i.e., construct the function

$$F(g) \stackrel{\rm def}{=} \int^g \frac{d\gamma}{\beta(\gamma)} \ , \tag{3.18}$$

and

3. Find the solution,

$$\overline{g}_{\rm RG}(x, g) = F^{-1}\{F(g) + \ln x\} \ . \tag{3.19}$$

4. Then the solution $\overline{g}_{\rm RG}$, which exactly satisfies the RG symmetry, i.e., is an exact solution of Eq. (2.1), corresponds to $\overline{g}_{\rm appr}$.

For illustration take the simplest perturbative expression for the effective coupling ,

$$\overline{g}_{\rm appr}(x, g) = g - \beta_1 g^2 \ln x \ . \tag{II $-$ 20pt}$$

Then the β-function will be of the form

$$\beta = -\beta_1 g^2 \ ,$$

and the solution is

$$\int_g^{\overline{g}} \frac{dg}{\beta(g)} = \frac{1}{\beta_1}\left(\frac{1}{\overline{g}} - \frac{1}{g}\right) = \ln x \quad .$$

i.e.

$$\overline{g}(x, g) = \frac{g}{1 + g\beta_1 \ln x} \qquad (\mathrm{II} - 20rg)$$

2 Application to classical problems

2.1 *Transfer problem*

. Here we consider an 1-dimentional transfer problem. The group equation for the transfer problem is

$$\frac{\partial \overline{G}(l, g)}{\partial l} = \beta(\overline{G}) \quad , \qquad \mathrm{I} - 8$$

and the generator

$$\beta(g) = \left.\frac{\partial \overline{G}(l, g)}{\partial l}\right|_{l=0} \quad .$$

is just the infinitesimal response at the boundary

$$\overline{G}(\epsilon, g) = g + \beta(g)\epsilon \quad .$$

Suppose we can find it from some simple reasoning (without solving the kinetic Boltzmann equation). Consider two simple cases when this response is linear (Case a) or quadratic (Case b) in the boundary density g,

$$\overline{G}(\epsilon, g) = g - \epsilon k g \quad \epsilon \ll 1 \quad (\text{Case a}) \quad ,$$

$$g - \epsilon \kappa g^2 \quad \epsilon \ll 1 \quad (\text{Case b}) \quad , \qquad (3.21)$$

Hence

$$B(g) = \Big\{ -kg \quad (\text{Case a}) \quad , -\kappa g^2 \quad (\text{Case b}) \quad . \Big\} .$$

and solving Eq.(I–8) we find

$$\overline{G}_{\mathrm{RG}}(l, g) = \Bigg\{ g e^{-kl} \quad (\text{Case a}) \quad , \frac{g}{1 + \kappa g l} \quad (\text{Case b}) \quad . \qquad (3.22)$$

These expressions possess the FSS property, i.e., they are solutions of Eq. (1.9). On the other hand, at small values of l they coincide with the input approximate expressions (3.21). They, however, describe solutions on the whole positive interval including the asymptotic region at $l \to \infty$, as it is shown at the Fig. III.1.

Figure III.1: Fig[II.1]

Figure III.2: Weak shock wave evolution

2.2 *Weak shock wave solution.*

Another example of RGM from classical physics is the 1-dimensional weak shock wave. When the wave arrived at a point l $(l = l_0 + \epsilon)$ as in Fig. III.2, the front velocity $V(\epsilon)$ takes the following form:

$$V(\epsilon) = V_0 - \frac{a}{2V_0}\epsilon \quad . \tag{3.23}$$

Hence the generator is

$$B(V) = -\frac{a}{2V} \quad . \tag{3.24}$$

Thus we find the solution,

$$V(l) = \sqrt{V_0^2 - a(l - l_0)} \quad . \tag{3.25}$$

which is well known in hydrodynamics.

Note that in this illustration we have a rather nontrivial nonperturbative example of β-function.

Here there stands an important and interesting problem, which has not yet been solved. It is the problem of transition to a "more strong" shock wave when viscosity becomes essential. There is no analytic solution like (3.25) and this case usually is analyzed by numerical calculations performed by computer. Here, the FSS = RG treatment may be "multi-coupling".

3 RGM usage in QFT

As it was explained above in §II.1 the QFT perturbation expression of finite order does not obey the RG symmetry. On the other hand in §II.4 it was shown that the 1-loop UV approximation for $\overline{g}$ used as an input in Eq. (3.17) for the construction of group generator $\beta(g)$ yields expression (20rg) that obeys the group symmetry and exactly satisfies the FEq.(1.6).

| We propose to the reader to check this fact. |

Now, using the geometric progression (20rg) as a hint let us represent 2-loop perturbative approximation for $\overline{g}$ in the form

$$\overline{g}_{\rm pt} = g - g^2\beta_1 \ln x + g^3\ [\beta_1^2 \ln^2 x - \beta_2 \ln x] + O(g^4)\ \ , \tag{3.26}$$

where β_1 and β_2 mean the β-function coefficients at the 1-loop and 2-loop level, respectively. If we substitute this expression into Eq.(1.6) we obtain

$$\overline{g}^{(2)}_{pt})x, g) - \overline{g}^{(2)}_{pt}(x/t; \overline{g}^{(2)}_{pt}(t, g)) = g^4\beta_1^3 \ln(x/t) \ln^2 t\ .$$

Meanwhile we can use $\overline{g}^{(2)}_{pt}$ as an input in Eq.(3.17). Now the step <u>1.</u> yields

$$\beta^{(2)}(g) = -\beta_1 g^2 - \beta_2 g^3$$

and then (step <u>2.</u>)

$$\beta_1 F^{(2)}(z) = -\int^z \frac{d\gamma}{\gamma^2 + b\gamma^3} = \frac{1}{z} + b \ln \frac{z}{1+bz}\ ; \quad b = \frac{\beta_2}{\beta_1}\ .$$

To make the last step we have to solve the equation for $\overline{g}^{(2)}_{rg}$:

$$F^{(2)}[\overline{g}^{(2)}_{rg}] = F^{(2)}(g) + \ln x \tag{3.27}$$

which, however, is a transcendental one and has no explicit solution. This means that we have to perform the step <u>3.</u> approximately. For this goal we can account that the second, logarithmic, contribution into $F^{(2)}(z)$ is a small correction to the first one at $bz \ll 1$. Under this reservation we can substitute 1-loop RG expression

$$\overline{g}^{(1)}_{rg}(l, g) = \frac{g}{1 + g\beta_1 l}\ \ , \qquad l = \ln z \tag{3.28}$$

instead of $\overline{g}^{(2)}_{rg}$ into this correction and obtain the explicit expression

$$\overline{g}^{(2)}_{rg} = \frac{g}{1 + \beta_1 gl + (\beta_2/\beta_1) g \ln\ [1 + g\beta_1 l]}\ \ . \tag{3.29}$$

This result (first obtained [3] in mid-fifties) is interesting in several aspects.

First, being expanded in g and gl powers, it produces an infinite series containing "leading", i.e., $\sim g^{n+1}l^n$, and "next-to-leading" ($\sim g^{n+2}l^n$) UV logarithmic contributions. Second, it contains nontrivial analytic dependence

$$\ln(1 + g\beta_1 l) \sim \ln(\ln Q^2)$$

which is absent in perturbation input. Third, being compared with Eq.(3.28) it demonstrates algorithm of subsequent accuracy improving, i.e., of RGM regilarity.

4 Essence of RGM

Now we can resume the RGM properties as follows. The RGM is a regular procedure of combining dynamical information (taken from approximate solution) with FSS group symmetry. The essence of RGM is the following:

1) The mathematical tool used in RGM is Lie differential equations.

2) The key element of RGM is possibility of (approximate) determination of group generators from dynamics.

3) The RGM works effectively in the case when a solution has a singular behaviour. It restores the structure of singularity. [It does not work at all in "trivial cases" when dynamical equation coincides with Lie equations.This is the case in our first macroscopic example with elastic rod .]

Part 2: Wilson RG

Chapter IV

DIFFERENT FACES of RENORM-GROUP

In 70's and 80's RGM was applied to (besides QFT and critical phenomena) polymers, turbulence, non-coherent radiation transfer, dynamical chaos, and so on. Simpler and physically transparent motivation in critical phenomena (than in QFT) makes this "explosion" of RG applications possible.

1 Wilson's Renormalization Group

1.1 *Critical phenomena*

RG in critical phenomena is based on Kadanoff-Wilson procedure [18], [7] referred to as "decimation" or "blocking". Consider a 2-dimensional spin lattice with spacing a, an elementary spin σ sitting at every site as in Fig. IV.1. The N, total amount of spin sites is supposed to be very large, of the Avogadro number order.

Figure IV.1: Lattice "blocking

The Hamiltonian describing the spin interaction of nearest neighbours is expressed as

$$H = k \sum_i \sigma_i \cdot \sigma_{i\pm 1} \quad . \tag{4.1}$$

where k is the coupling constant.

The statistical sum is obtained from the partition function,

$$S =< \exp(-H/\theta) >_{\text{average}} \quad .$$

To realize the blocking or decimation, one has to perform an "averaging" over blocks consisting of n elementary sites. This is a very essential step as far as it diminishes the degree of freedom number (from N to N/n). It destroys the small-range properties of the system under consideration, in the averaging course some information being lost. However, the long-range physics (like critical phenomena of phase transition) is not affected by it and we gain the simplification of our problem. After this procedure new

Figure IV.2: Effective lattice

effective spins $\mathbf{\Sigma}$ arise on new effective lattice as shown in Fig. IV.2 where the $n = 3$ and $n = 4$ blocking are illustrated. We obtain also the new effective Hamiltonian,

$$H_{\text{eff}} = K_n \sum_I \mathbf{\Sigma}_I \cdot \Sigma_{I\pm 1} + \Delta H \quad ,$$

where ΔH contains quartic and higher terms,

$$\Delta H = \sum \mathit{\Sigma\Sigma\Sigma\Sigma} + \cdots \quad .$$

For IR properties (long-distance), ΔH is unessential. Hence we can conclude that the spin averaging leads to a (approximate) transformation,

$$k \sum_i \sigma \cdot \sigma \to K_2 \sum_I \Sigma \cdot \Sigma \quad , \tag{4.2}$$

or, taking into account the size of "elementary block" change, generally

$$KW_n \ : \ \left\{ a \to \sqrt{n}a \ \ , k \to K_n \ \ . \right\} .$$

The latter is called Kadanoff-Wilson transformation.

In general the "new" coupling constant K_n is a function of the "old" one and the decimation number n. It is convenient to write it down in the form $K_n = \bar{K}(1/n, K)$. Then we can formulate the KW transformation as follows

$$KW(n) : \left\{ a \to \sqrt{n}a, \quad k \to K_n = \bar{K}\left(\frac{1}{n}, k\right) \right\} \quad . \tag{4.3}$$

These transformations obeys the group composition law

$$KW(n) \cdot KW(m) = KW(nm)$$

if

$$\bar{K}(x,k) = \bar{K}(\frac{x}{t}, \bar{K}(t,k)) \quad \left[x = \frac{1}{nm}, t = \frac{1}{n} \right] \quad , \tag{4.4}$$

This is just a RG = FSS symmetry.
We observe the following points:

- The FSS symmetry is approximate (due to ΔH).
- The transformations $KW(n)$'s are discrete.
- There exist no reverse transformation.

Hence Kadanoff-Wilson RG is an approximate and discrete semi-group. In a long distance sence (IR limit), however, ΔH is irrelevant, $\Delta(1/n)$ is close to continuum and it is possible to use differential Lie equations.

In application of these transformations to the critical phenomena the notion of a *fixed point* is important. As explained below, in the Section IV.1, it is usually associated with power type asymptotic behavior. Note here that, contrary to the QFT case considered in Section IV.1, in phase transition physics we deal with IR stable points.

1.2 *Polymer theory*

In the polymer physics one consider statistical properties of polymer macromolecules that can be imagined as a very long chain of identical elements. The number of elements N, called the polymerization rate, could be as big as 10^5, the macromolecular size reaching several hundred Angstroms.

Such a big molecular chain form a specific pattern resembling the pattern of random walk. Indeed these two problems can be formulated in a similar way. The central problem of polymer theory is very close to that of the random walk and can formulated as follows.

Figure IV.3: Polymer chain

Take a very long chain of N steps (the length of each "step" $= a$) as in Fig. IV.3 and define the "chain size" R_N , i.e., the distance between the "start" and the "finish" points, the distribution function of angles ϕ_i between neighbring elements being given. This function $f(\phi)$ is defined by the forces between adjacent elements depending on some external factors like temperature T. The essential feature of polymer chain is the impossibility of self-intersection. This is known as an accounting *excluded volume* effects.

In reality polymer molecules are swimming in a solvent and form *globulars* - see Fig. IV.4.

Figure IV.4: Globulars ...

For large N values the molecular size R_N follows the power low

$$R_N \sim N^\nu \qquad [\nu = \text{Fleury index}] \; . \tag{4.5}$$

– the Fleury law. When n is given, R_N is a function of $f(\phi)$ which depends on external conditions (e.g., temperature T, properties of solvent, *etc.*). If T increases, R_N increases and at some moment globulars become touch one another. This is the polymerization very similar to a phase transition phenomena.

The RG ideology was introduced in polymer physics by De Gennes [19], [20]. The key idea was the grouping of n chain subsequent elements into new "elementary blocks" as is shown in Fig. IV.5.

Figure IV.5: Blocking in polymers

This grouping operation is very close to Kadanoff's blocking. It leads to the transformation,

$$\left\{ \begin{array}{l} 1 \to n \\ a \to A_n \end{array} \right\} \quad . \tag{4.6}$$

which is analogous to one for spin lattice decimation. The second transformation must be specified by direct calculation that gives the explicit form of $A_n = \bar{a}(n,a)$. Here we have a discrete semi-group. When, by using RG technique one finds the fixed point then one obtains the Fleury power law (4.5) and can calculate its index ν.

Generally the excluded volume account yields some complifications. However inside thr RGM framework it can be made rather simply [13] by introducing additional argument similar to finite length L in elastic rod or transfer problem and particle mass m in QTF.

1.3 *Noncoherent radiation transfer*

Here we consider propagation of light quanta in a media with a specific property. The cross-section for the scattering of a quantum on atom of a media has a narrow and very high Breit-Wigner resonance with value at maximum σ_{max} being much higher than the cross-section "background" value σ_0 outside the resonance region:

$$\sigma_{\max} \gg \sigma_0 \sim \text{background} \ . \tag{4.7}$$

as in Fig. IV.6. Suppose further that the background mean free path $l_0 \sim \sigma_0^{-1}$ is rather

Figure IV.6: Transfer ...

large and comparable with the size of the media. Then, the given quantum "travel history" can be modelled by a Monte-Carlo simulation method with a reasonable total number of collisions. However, a photon moving through the media also "walks" in frequency due to heat motion and interactions of gas molecules.

Due to this, starting with some background frequency value it can happen to enter into resonance zone with very small mean free path

$$\ell_{\rm r} \sim \frac{1}{\sigma_{\rm max}} \quad . \tag{4.8}$$

This leads to a specific trouble. The method of successive collisions ceases to be effective as far as the most part of the computer time is "eated" to pass this dangerous resonance neigboroughood region as shown in Fig. IV.7. The "blocking idea" [21], [22] in this case

Figure IV.7: resonance...

is

- To change the Breit-Wigner curve into the step function with the width Γ.
- To calculate separately the effective walk distance across the dangerous zone as a function of Γ: $\ell(\Gamma)$.

Thus we have a pair $\{\Gamma,\ \ell(\Gamma)\}$, and can scale and apply the RG ideology and equations [13,14] ???. The resulting picture is illustrated by Fig. IV.8. ??

Figure IV.8: Blocking in non-coherent transfer...

2 Turbulence

The formalism of the turbulence problem with the RG ideology is obtained [23], [24], [25] along the following steps:

1. Define the generating functional.
2. Write the path integral representation.
3. Find the equivalence of the classical statistical system to some quantum field theory.
4. Construct the system of Schwinger-Dyson equations.
5. Apply the Feynman diagram technique.
6. Perform the finite renormalization procedure.
7. Then the RG ideology and equations are obtained.

The physics of the turbulence problem is displayed in Fig. IV.9.

3 Paths of RG expansion

We see that RG is expanded in different fields of physics by two different ways:

- the direct analogy with Kadanov-Wilson construction (averaging over some set of degrees of freedom) on polymers, non-coherent transfer and percolation, i.e., constructing a set of models for a given physical problem.

Figure IV.9: turbulence..

- search for the exact symmetry (FSS = RG) by proof of the equivalence with QFT: e.g., on turbulence (de Dominicis and Martin [23]; A. Vasiliev *et al.* [25]), plasma turbulence [27] and phase transition (continuous spin-field models by K. Wilson [7]).

Are there different renormalization groups? – The answer is YES:

1. In QFT and macroscopic examples, RG $\equiv$ FSS symmetry is

   an exact symmetry of the solution formulated in its natural variables.

2. In turbulence, continuous spin-field models and some others, it is

   a symmetry of an equivalent QFT model.

3. In polymers, percolation, *etc.* , (with Kadanov-Wilson blocking), the RG transformation is a transformation between different auxilliary models (specially constructed for this purpose).

As we have seen there is no essential difference in mathematical formalism. There exists, however, a profound difference in physics:

A. In some cases (QFT and macroscopic examples), the FSS = RG is an exact symmetry of **a solution**.

B. In Kadanov-Wilson type problem (critical phenomena, polymers, *etc.*), one has to construct a set $\mathcal{M}$ of models M_i as in Fig. IV.10. The RG transformation is

$$R(n)M_{(i)} = M_{(n\cdot i)} \quad , \tag{4.9}$$

Figure IV.10: Wilson transformations

acting inside of the set of models. Thus the FSS symmetry "exists" only inside this set of models.

Let us also make a comment about the mathematical difference between different renormalization groups. In the QFT, as well as in macroscopical examples given above in the Part 1, RG (or FSS) is an exact continious symmetry group. On the other hand in critical phenomena, polymers and some other similar cases (with *averaging* operation) it is approximate, discrete semi-group. We should add that dynamical chaos theory where some people use RG ideas and terminology (see, e.g., ref [28] the functional iterations does not form any group at all.

4 Two faces of RG in QFT

An important notion in QFT is the invariance with respect to RG transformation, i.e., renorm-invariance. This implies that the results are independent of the choice of renormalization scheme. The symmetry underlying the RG transformations in QFT is exact. In this case the running parameter q being the logarithm of a 4-momentum squared and g, the renormalized coupling constant.

An essential feature of QFT is the presence of virtual states and virtual transitions. In QED, for example, the processes of virtual transition of a photon into an electron-positron pair and vice versa can take place. This process of vacuum polarization leads to several specific phenomena and particularly to the notion of effective charge. In classical electro-magnetism, an external electric charge in the polarizable medium attracts opposite sign charged particle in the medium so that the charge is partially screened. In QFT, the vacuum, i.e., the interparticle space itself, stands for "polarizable medium". Due to the vacuum fluctuation the charge is screened. It was first shown by Dirac (1933) [14] in position/momentum space. Q^2 dependence of an electron charge is given by

$$e(Q^2, \Lambda^2) = e_0 \left\{ 1 + \frac{\alpha_0}{6\pi} \ln \frac{Q^2}{\Lambda^2} + \ldots \right\} \quad . \tag{4.10}$$

expressed in terms of bare electron charge $e_0 = \sqrt{\alpha_0}$ and of momentum cutoff Λ.

The first approach of using RG idea to this problem was done by Stüeckelberg and Peterman [1]. In their pioneering investigation the RG transformation was introduced rather formally within the renormalization of infinities procedure the result of which contains finite arbitrariness. Just this degree of freedom" in finite renormalized expressions was used by Bogoliubov and Shirkov [2] – [5]. Roughly speaking this corresponds to the change ($\Lambda \to \mu$) of the *degree of freedom parameter*. Instead of (13) we have now

$$e(Q^2, \mu^2) = e \left\{ 1 + \frac{\alpha}{6\pi} \ln \frac{Q^2}{\mu^2} + \ldots \right\} \quad . \tag{4.11}$$

Another approach was used by Gell-Mann and Low [6]. Their paper was devoted to the short distance behaviour in QED and the "Λ degree of freedom" was used to analyze the UV behaviour.

In the first approach [1], [2], the effective charge is given by a "finite representation"

$$\bar{\alpha}(Q^2) = \alpha_\mu \left(1 + \frac{\alpha_\mu}{3\pi} \ln \frac{Q^2}{\mu^2} + \cdots \right) \quad . \tag{4.12}$$

Here, the FSS = RG symmetry is formulated in terms of Q scale and the μ represents the reference point as shown at Fig. IV.11. On the other hand, in the second approach [6]

Figure IV.11: Fig[IV.11]

the charge is given by the "singular representation"

$$\bar{\alpha} = \alpha \left(1 + \frac{\alpha}{3\pi} \ln \frac{Q^2}{\Lambda^2} + \cdots \right) \quad . \tag{4.13}$$

We can draw a physically simple picture (as was commented later by Wilson in his Nobel lecture) to this approach. Effectively we can imagine the electron of a finite size, smeared over a small volume with radius $R_\Lambda = \hbar/c\Lambda$, $\ln(\Lambda^2/m_e^2) \gg 1$. The electric charge $e(\Lambda)$ of such a non-local electron is considered as depending on the cut-off momentum Λ so that this dependence accumulates the vacuum polarization effects which takes place at distances from the point electron smaller than R_Λ. We have the set of models with non-local electron correponding to the values of the cut-off Λ as shown in Fig. IV.12.

Figure IV.12: QED duplicity

Here, α_N depends on R_N and the vacuum polarization effects in the excluded volume R_N^3 should be subtracted. In this language the RG transformation is the transition from one value of the smearing radius to another $R_i \to R_j$, simultaneously with a corresponding change of the effective electron charge $e(\Lambda_i) \to e(\Lambda_j)$. In other words, RG symmetry here is the symmetry of operations in the space of models of non-local QED constructed in such a way that at large distances every model is equivalent to the real local one. It is correct to say that the Renormalization Groups in two approaches are different from each other.

5 Summary

Before going into the discussion of some applications of RG in QFT, we give the summary of the general properties of RG and FSS. The RG = FSS symmetry is not a symmetry of a physical system, but it is a symmetry of solution with respect to changing a way of parametrization, in terms of a natural quantity, or it is a symmetry inside a set of models exploiting the way of their (i.e. models) parametrization.

The RG Method is a regular method of improving approximate solution by its "Marriage" with group symmetry. The RGM is effective in the case of solution with singularity. The RG Approach and RGM are general powerful methods in theoretical and mathematical physics.

Part 3: RG in QFT

Chapter V

RENORM-GROUP in QFT. General

This chapter being devoted to RG application in QFT, mainly to UV asymptotic behavior, is oriented on particle physicists. We discuss first the specific features of UV analysis connected with perturbation theory use in its practical applications. Then we consider problems of scheme and gauge dependence.

1 UV analysis in general

1.1 *One-coupling case*

. General analysis of UV asymptotic behavior for the one-coupling QFT model can be performed rather simply on the basis of solution of massless group equation for the effective coupling

$$\int_g^{\bar{g}(l)} \frac{d\gamma}{\beta(\gamma)} = l \equiv \ln \frac{Q^2}{\mu^2} \quad , \tag{5.1}$$

where $g = \bar{g}(0)$ (i.e. $\bar{g}$ at $Q^2 = \mu^2$) and μ is a renormalization point. As follows from it, the asymptotic at $\ln x \to \infty$ corresponds to the divergence at the upper limit of the l.h.s. integral. Depending on the behaviour of the β function shown in Fig. 1.1, the resultant UV behaviour of the coupling $\bar{g}$ differs very much.

Suppose first that at very small g values beta-function is positive. Then three cases are possible :

a) Consider first the situation when

$$\int_g^{g_*} \frac{dz}{\beta(z)} = \infty \tag{5.2}$$

corresponding to the case when the beta-function has a zero at g_* as shown in Fig. V.2. Here the UV asymptotic value of effective coupling is finite

Figure V.1: Relation btwn β-funcrtion and UV ...

Figure V.2: Finite fixed point

$$\bar{g}(\infty, g) = g_* < \infty. \tag{5.3}$$

that corresponds to the finite renormalization of the coupling constant.

$$Z = \bar{g}(\infty, g)/g.label5.4 \tag{5.4}$$

Using the qualitative theory of DEs terminology one can say that at $g = g_*$ we have a UV *fixed point.*

If we admit that at $g = g_*$ there is a first order zero

$$\beta(g) \simeq b(g_* - g) \tag{5.5}$$

then eq. (5.2) gives

$$\bar{g}(l, g) - g_* \simeq C \exp^{-bl} = C(\frac{Q^2}{\mu^2})^{-b} \quad \text{as} \quad Q^2 \to \infty$$

i.e., in the vicinity of a fixed point we have asymptotic power regime.

b) If $\beta(g)$ is (monotonically) increasing but gentler than g^2, so that

$$\int_g^\infty \frac{dz}{\beta(z)} = \infty$$

Figure V.3: Infinite renormalization ...

then the effective coupling tends to infinity as shown in Fig. V.3.

$$\lim_{x\to\infty} \bar{g}(x,g) \to \infty \tag{5.6}$$

that corresponds to infinite coupling constant renormalization. Formally this is equivalent to the $g_\infty = \infty$.

c) At

$$\int_g^\infty \frac{d\gamma}{\beta(\gamma)} = L = \ln x_\infty < \infty$$

that happens if

$$\lim_{x\to\infty} \beta(g)/g^2 \geq \text{const}$$

the theory has an inner contradiction, as far as

$$\bar{g}(x_\infty, g) = \infty \quad \text{at} \quad x_\infty < \infty$$

and the momentum region $x > x_\infty$ can not be described by the theory. We encounter here with a *ghost trouble* as explained below in the Section 2.2 and illustrated in Fig. V.4. Up to now we assumed that the generator $\beta(g)$ is positive. In the opposite case

Figure V.4: "Ghost-trouble" case

d) $\beta(g) = -b(g) < 0$ one has to consider the equation

$$\int_{\bar{g}}^{g} \frac{dz}{b(z)} = \ln x$$

and study possible divergence of the integral involved on the lower limit.

If this occurs at some finite value $g = g_\infty$ then the situation is quite analogous to the case *b*) considered above and described by Eqs.(5.6),(5.4). The only difference is that now the effective coupling tends to its limiting value g_∞ from above.

As the most important case we consider the possibility when the singularity lies at the origin :

$$\beta(0) = 0$$

That happens in QCD. Then in the UV limit $\bar{g}$ vanishes $\bar{g}(\infty, g) = 0$ that corresponds to the *asymptotic freedom* phenomenon.This situation is illustrated by Fig. V.5. If we

Figure V.5: Asymptotic freedom case

assume here that $\beta(g) = -bg^2$ at $g \to 0$ then

$$\bar{g}(x, g) \to \frac{1}{b \ln x} \quad \text{as} \quad x \to \infty \tag{5.7}$$

1.2 *Multi-coupling case*

For the quantum field model with several coupling constants one has to consider the system of coupled functional or differential Eqs. (I–14), (II–10). The last ones can be analyzed by the well known methods of the qualitative theory of nonlinear differential equations. Take the case when there are two coupling constants g and h. The system of evolution differential equations is of the form

$$\dot{\bar{g}} = \beta_g(\bar{g}, \bar{h}), \qquad \dot{\bar{h}} = \beta_h(\bar{g}, \bar{h}). \tag{5.8}$$

with $\dot{f} = \partial f / \partial \ln x$. According to Eq.(3.16?) the general solution of this system can be written in the form

$$F(\bar{g}, \bar{h}) = F(g, h) + l\ , \qquad \Phi(\bar{g}, \bar{h}) = \Phi(g, h) + l\ ; \quad l = \ln x. \tag{5.9}$$

where F and Φ - two arbitrary reversible functions. By differentiating these Eqs. with respect to l we obtain

$$\beta_g(g,h)\dot{F}_1(g,h) + \beta_h(g,h)\dot{F}_2(g,h) = 1,$$

$$\beta_g(g,h)\dot{\Phi}_1(g,h) + \beta_h(g,h)\dot{\Phi}_2(g,h) = 1.$$

As far as argument $l = \ln x$ does not enter explicitly into generators β and b, it can be formally excluded by dividing one equation on the other :

$$\frac{d\bar{g}}{d\bar{h}} = F(\bar{g},\bar{h}), \qquad F = \frac{\beta_g}{\beta_h}$$

This equation can be analyzed on the two dimensional phase plane $(\bar{g},\bar{h})$ shown in Fig. V.6. We shall present an explicit example of such phase portrait below at the end

Figure V.6: Phase plane ...

of the Section 3.2 (see Fig. VI.5).

The essential features now are singular points and singular solutions. Singular points correspond to $\beta_i = 0$ (or ∞). They can be of different types: stable fixed point that is known as *attractor*, unstable fixed point and saddle-type fixed point. In the vicinity of UV attractor one can have a power scaling behavior as in Eq.(5.5). Singular solutions join singular points and also can be stable or unstable. In our case the unstable ones separate the parts of phase plane with different UV asymptotes.

2 Perturbative approach to the UV asymptote

2.1 *Structure of RG results*

Consider a general situation with RG approach to the UV asymptotic behavior based on perturbation calculations input. In a one-coupling QFT case group generators entering into differential equations Eq.(5.3), perturbatively, can be written as

$$\beta(g) = \beta_1 g^2 + \beta_2 g^3 + \dots, \qquad \psi(g) = \psi_1 g + \psi_2 g^2 + \dots .$$

Generally, expansion coefficients depend on mass variable

$$\beta(y,g)=\sum_{l\geq 1}\beta_1(y)g^{l+1},\qquad \psi(y,g)=\sum_{l}\psi_l(y)g^l. \tag{5.10}$$

Note, that if g is just the expansion parameter (that can be equal to coupling constant or to its square) then, usually, the first term in perturbation expansion for β is quadratic and for ψ - linear as is explicitly indicated in (1) and (2).

Substituting (1) in (5.3), and reexpanding the ratio $\gamma^2/\beta(\gamma)$ we obtain after integration

$$1/g-1/\bar{g}-\frac{\beta_1}{\beta_2}\ln(\bar{g}/g)-b_3(\bar{g}-g)+0(\bar{g}^2,g^2)=\beta_1\ln x, \tag{5.11}$$

$$\ln s(x,g)=(\psi_1/\beta_1)\ln\{\bar{g}(x,g)/g\}+c_2(\bar{g}-g)+0(g^2), \tag{5.12}$$

$$b_3=\beta_3/\beta_1-(\beta_2/\beta_1)^2,\quad c_2=[\psi_2/\beta_1](\psi_2/\psi_1-\beta_2/\beta_1). \tag{5.13}$$

As follows, solutions $\bar{g}$ and s depend on two arguments g and $g\ln x$. By expanding in powers of g we get

$$\bar{g}(x,g)=gf_1(g\ln x)+g^2f_2(g\ln x)+\ldots, \tag{5.14}$$

$$\ln s(x,g)=\varphi_1(g\ln x)+g\varphi_2(g\ln x)+\ldots, \tag{5.15}$$

where f_j and φ_i can be obtained in explicit form. For example

$$f_1(z)=\frac{1}{1-\beta_1 z},\qquad \varphi_1(z)\sim f_2(z)\sim\ln f_1(z). \tag{5.16}$$

Comparing expressions (5.5) with usual perturbative expansions

$$\bar{g}_{pt}(x,g)=g+g^2\beta_1\ln x+g^3[\beta_1^2\ln x+\beta_2\ln x]+0(g^4),$$

$$s_{pt}(x,g)=1+g\psi_1\ln x+g^2[(\beta_1\psi_1/2)\ln^2 x+\psi_2\ln x]+0(g^3)\ ,$$

used as input for obtaining our starting generators in the form Eqs. 5.1), one can see the qualitative effect of RGM using. In the case considered it changes the region of applicability of perturbation method initially limited by the condition

$$g\ln x\ll 1$$

to the more large region defined now by two relations

$$g\ll 1,\qquad \bar{g}(x,g)\ll 1,$$

the second of which is defining.

2.2 *The Ghost-Pole Trouble*

Turn now to the one-loop RG approximation for the effective coupling $\bar{g}$, considered in the UV, i.e., massless, limit. According to eqs.(5.4), (5.5) it has the form

$$\bar{g}_{(1)}(x,g) = \frac{g}{1-\beta_1 g \ln x} \ .$$

Suppose now that the numerical coefficient β_1 is positive. Such is the case, e.g., in QED where $\beta_1 = 1/3\pi$ and g stands for expansion parameter $\alpha = e^2/4\pi$. This expression obviously has a pole singularity at

$$x = x_* \equiv \exp(1/\beta_1 g) = \exp(3\pi/\alpha).$$

As far as QED effective coupling is proportional to (transverse part of) photon propagator such a pole generally can correspond to some bound state of a system with photon quantum numbers. However the pole describing a physical bound state must have a positive residue, while the l.h.s. of Eq.(5.9) has a negative one.

This means that it corresponds not to physical but rather to some *ghost* bound state. The presence of such unphysical singularity can be considered as a signal of a theory inconsistency. Such claims have been made in the midst of 50-ies when the ghost-pole trouble was first discovered in QED just before the creation of RG method.

This method turns out to be very effective for general discussion of the ghost-pole trouble. The first question that must be answered here is the stability of result of ghost-pole existence with respect to multi-loop correction account.

3 Scheme Dependence

3.1 *Effective and invariant couplings*

Renormgroup formalism in QFT is of a most simple structure for the case when renormalization subtraction in vertices and propagators forming invariant coupling is performed at some definite values of momentum variables, i.e., in a MOM-scheme. Here, the normalization condition for the invariant coupling is of the simple form

Invariant Coupling$(Q^2 = \mu^2)$ = coupling constant (2.9)

while one-argument functions, like propagators, are normalized to unity. Just in this case group functional and differential Eqs. are of the form given above.

However, condition (2.9) is not generally fulfilled, e,g, in the minimal subtraction (MS) scheme. Consider the case when the normalization condition for invariant coupling $G(x,y;g)$

$$G(1,y;g) = Q(y,g) \tag{5.17}$$

contain some function Q , which can be presented in the form

$$Q(y,g) = g + Q_1 g^2 + Q_2 g^3 + \dots \tag{5.18}$$

with coefficients depending on the mass argument: $Q_l = Q_l(y)$. The function reverse to Q with respect to the second argument

$$q(y,g) = Q^{-1}_{(2)}(y,g), q\{y, Q(y,g)\} = g$$

will also be useful. Perturbatively,

$$q(y,\gamma) = \gamma + q_1\gamma^2 + q_2\gamma^3 + q_3\gamma^4 + \ldots =$$

$$\gamma - Q_1\gamma^2 - (Q_2 - 2Q_1^2)\gamma^3 - (Q_3 - 5Q_2Q_1 + 5Q_1^3)\gamma^4 + \ldots \quad (5.19)$$

Note here that we have introduced new special notation G for invariant coupling (IC) with an arbitrary normalization function Q to distinguish it from an effective coupling (EC) normalized to g for which in the following we keep the notation $\bar{\mathrm{g}}$.

Functional equation for IC has the form

$$G(x,y;g) = G\{x/t, y/t; q[y/x, G(t,y;g)]\} \quad (5.20)$$

As can be easily checked, this relation obeys group property and at the same time corresponds to the normalization condition (2.9).

By differentiating (5.20) in two various ways we obtain two differential Eqs .

$$\frac{\partial G(x,y;g)}{\partial \ln x} = B(\frac{y}{x}, G(x,y;g)), \quad (5.21)$$

$$\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g}\}G(x,y;g) = 0 \quad (5.22)$$

with

$$B(y,g) = \frac{\partial}{\partial \xi}G\{\xi, y; q(y,g)\} \quad at \quad \xi = 1 \quad (5.23)$$

and

$$\beta(y,g) = \{B(y,\gamma)\frac{\partial}{\partial \gamma} - y\frac{\partial}{\partial y}\}q(y,\gamma)|_{\gamma=Q(y,g)} \ . \quad (5.24)$$

Note here that for IC we have two different beta-functions entering in Eqs.(5.21), (5.22) and related by (5.24). It turns out to be possible to treat invariant coupling G as an effective coupling corresponding to the "new" coupling constant

$$\gamma = Q(y,g) \quad (5.25)$$

Defining the new object

$$\bar{\gamma}(x,y;\gamma) = G\{x,y;q(y,\gamma)\} \quad (5.26)$$

we see that it satisfies the group functional Eq. of a normal type

$$\bar{\gamma}(x,y;\gamma) = \bar{\gamma}\{x/t, y/t; \bar{\gamma}(t,y;\gamma)\},$$

the differential Eq. with generator B and can be considered as EC corresponding to γ. On the other hand, if we define new auxiliary function $\tilde{g}$ by the relation

$$\tilde{g}(x,y;g) = q\{y/x,;G(x,y;g)\} \tag{5.27}$$

it will satisfy the "normal" functional Eq.

$$\tilde{g}(x,y;g) = \tilde{g}\{x/t,y/t;\tilde{g}(t,y;g)\} \tag{5.28}$$

as well as the normalization condition (2.9) and can be considered on equal footing with EC. We shall refer to it as to the "group effective coupling" to stress that generally it is not related to IC by the Dyson transformation. Note, that both group differential Eqs. for $\tilde{g}$ contain the same generator β .

3.2 *Scheme Dependence*

In the perturbative QFT calculations explicit expressions for renormalised vertices and propagators are obtained in the form of power expansions with coefficients depending on the renormalisation scheme employed. In our context it is essential : does in the given renormalisation scheme the IC coincide with *EC* or it does not. Generally, they are different and, as we have shown, considered inside the same renormalisation scheme can be related by the "transition function" Q depending on mass variable (see Eqs. (5.25) and (5.26) above).

Consider now the question of correspondence between results of calculations performed in two different renormalisation schemes. Suppose in each of them IC = EC (e.g., - in two different MOM schemes). Let respective coupling constants g and γ be related by the equation

$$\gamma = Q(y,g), g = q(y,\gamma). \tag{5.29}$$

Then according to the relations (5.25), (5.26) effective couplings will be related by

$$\bar{\gamma}(x,y;\gamma) = Q\{y/x,\bar{g}(x,y;g)\} \tag{5.30}$$

Note also that corresponding group generators are connected by the relation (5.24).

3.3 *Renormalisation ambiguities*

It was mentioned that effective coupling (as well as an invariant coupling) for a given QFT model is not a single-valued notion. Corresponding ambiguity, generally, can be related to the coupling constant transformation of a specific form

$$g \to \gamma = g + Q_1 g^2 + Q_2 g^3 + \ldots = Q(g) \tag{5.31}$$

(considered above in the Section 3c.) which arises, e.g., under change of normalization momenta. Remind here that in mess-dependent case the normalization condition for invariant coupling $G(x,y;g)$ is of the following form

$$G(1,y;g) = Q(y,g), \tag{5.32}$$

Q being the function of the type (5.31) with Q_l depending on y. Correspondingly, the group functional Eq. for IC has a rather complicated form {see Eq. (3.20)} and differential Eqs.(3.21,22) contain two different beta-functions related by (3.24)

$$\beta(y,g) = B(y,\gamma)\frac{\partial q(y,\gamma)}{\partial\gamma}\bar{)} - y\frac{\partial q(y,\gamma)}{\partial y} \quad \text{at} \quad \gamma = Q(y,g) \tag{5.33}$$

At the same time IC G can be treated as an EC for the new coupling constant g introduced in (5.31) with the definition

$$\bar{\gamma}(x,y;\gamma) = G(x,y;q(y,\gamma)). \qquad 3.26 \tag{5.34}$$

On the other hand the group effective coupling (GEC) defined as

$$\tilde{g}(x,y;g) = q\{y/x, G(x,y;g)\} \qquad 3.27 \tag{5.35}$$

generally cannot be related to G by renormalization transformation. Differential Eqs. for $\tilde{g}$ will contain $\beta(y,g)$ as a generator.

Note also that it is possible to consider the transition

$$g \to \gamma = Q(y,g) \tag{5.36}$$

as a coupling constant transformation related to the change of renormalization prescription. In other words, starting from the set

$$\{g, \bar{g}(x,y;g), \beta(y,g)\}$$

we can arrive at the another one

$$\{\gamma, \bar{\gamma}(x,y;\gamma), B(y,\gamma)\}$$

by the transformation (5.36) and transition

$$\bar{g} \leftarrow \bar{\gamma}(x,y;\gamma) = Q\{y/x, \bar{g}\{x,y;q(y/x,\gamma)\}\} \tag{5.37}$$

which is a composition of (5.34) and (5.35) with generators b, B related by Eq.(5.33).

Quite analogously in the group functional Eq. for an one-argument (like, e.g., propagator amplitude) function $s(x,y;g)$ it is nesessary not only to take into account relation (5.35) between IC G and GEC $\bar{g}$

$$s(x,y;g) = \frac{s(t,y;g)}{s\{1,y,\tilde{g}(t,y;g)\}} s\{x/t, y/t; \tilde{g}(t,y;g)\} \tag{5.38}$$

but also normalization of s itself

$$s(1,y;g) = S(y,g) = 1 + s_1(y)g + s_1(y)g^2 + \dots .$$

The corresponding differential Eqs. have the form

$$\frac{\partial \ln s(x,y;g)}{\partial \ln x} = \gamma[\frac{y}{x}, \tilde{g}(x,y,g)] \tag{5.39}$$

and

$$\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(x,y;g)\frac{\partial}{\partial g}\right] \ln s(x,y;g) = \tilde{\gamma}(y/x,g) \tag{5.40}$$

with

$$\gamma(y,g) = \frac{\partial}{\partial t} \ln s(t,y;g) \quad \text{at} \quad t = 1, \tag{5.41}$$

$$\tilde{\gamma}(y,g) = \gamma(y,g) + \{y\frac{\partial}{\partial y} - \beta(y,g)\frac{\partial}{\partial g}\} \ln S(y,g) \tag{5.42}$$

In a given QFT model with one coupling constant EC and IC are fixed by the choice of renormalisation scheme specified by the way of vertex subtraction. However, in the case when the same coupling constant g enters in several Lagrangian structures, like in QCD, for definition of G different vertices can be used. This means that in such a case we have several ICs. The corresponding ambiguity can be described by transformations just considered. We shall return back to this point below in Chapter VI.

3.4 *Perturbation Theory*

Expanding generators β and B in powers of the coupling constants

$$\beta(y,g) = \beta_1 g^2 + \beta_2 g^3 + \beta_3 g^4 + \beta_4 g^5 \ldots, \tag{5.43}$$

$$B(y,\gamma) = B_1\gamma^2 + B_2\gamma^3 + B_3\gamma^4 + B_4\gamma^5 \ldots \tag{5.44}$$

with the help of Eq. (3) it is possible to obtain relations between the expansion coefficients

$$\begin{gathered} B_1 = \beta_1 + q_1'(y), B_2 = \beta_2 + q_2'(y) - 2q_1'(y)q_1(y), \\ B_3 = \beta_3 + q_1\beta_2 + (q_1^2 - q_2)\beta_1 + F_3\{q_i', q_j\}, \\ B_4 = \beta_4 + 2q_1\beta_3 + q_1^2\beta_2 + [-q_3 + 2q_1q_2 - q_1^3]\beta_1 + F_4\{q_i', q_j\}, \end{gathered} \tag{5.45}$$

where $q_i'(y) = dq_i(y)/d\ln y$ and

$$F_3 = q_3' - 2q_2'q_1 - 3q_2q_1' + 4q_1'q_1,$$

$$F_4 = q_4' + (4q_1^2 - 3q_2)q_2' - 2q_3'q_1 + (-4q_3 + 12q_2q_1 - 8q_1^3)q_1'.$$

Analogous relations for "anomalous dimensions" coefficients are

$$G_1 = g_1 + s_1', \quad G_2 = g_2 - q_1g_1 + \{s_2' - s_1's_1\}, \ldots. \tag{5.46}$$

As follows from Eqs.(15) and (16) variations of the first two coefficients of a beta-function as well as the first coefficient of g are connected with mass dependence of the "scheme changing" functions Q and S. This means that β_1, β_2 and γ_1 are invariant under mass independent coupling constant transformation.

3.5 *Multi-coupling case*

For a more general situation with several coupling constants corresponding relations are similar, but it must be noticed that here in a massless case only first (1-loop) coefficients of generator expansions are scheme independent.

Let us start with two invariant couplings G and H that correspond to coupling constants g and h. Suppose that normalization conditions have the form

$$G(1,y;g,h)=\alpha(y,g,h)\ ,\qquad H(1,y;g,h)=\gamma(y,g,h). \tag{5.47}$$

Consider the simultaneous transformation of two expansion parameters g and h (i.e. renormalized coupling constants possibly squared):

$$g\to\alpha=\alpha(y,g,h), h\to\gamma=\gamma(y,g,h). \tag{5.48}$$

Here it should be mentioned that in the weak coupling case thansformation functions can be presented by expansions:

$$\alpha=g+a_{20}g^2+a_{11}gh+a_{02}h^2+a_{30}g^3+a_{21}g^2h+a_{12}gh^2+\dots \tag{5.49}$$

$$\gamma=h+c_{20}h^2+c_{11}gh+c_{02}g^2+c_{30}h^3+c_{21}gh^2+c_{12}g^2h+...$$

with coefficients a_{ij}, c_{kl} depending on mass argument. Note here the presence of terms like $a_{02}h^2, c_{02}g^2, c_{03}g^3$, that can be considered as nontrivial.

Invariant couplings G and H satisfy the system of coupled functional equations

$${}^{G}_{H}(x,y;g,h)={}^{G}_{H}\{\frac{x}{t},\frac{y}{t};A(y/t,G_t,H_t),\Gamma(y/t,G_t,H_t)\}, \tag{5.50}$$

with

$$G_t=G(t,y;g,h),\quad H_t=H(t,y;g,h)$$

and A,Γ being the functions reverse to α,γ with respect to second and third arguments, i.e.;

$$\alpha(y,g,h)=\alpha,\gamma(y,g,h)=\gamma \text{ for } g=A(y,\alpha,\gamma),h=\Gamma(y,\alpha,\gamma)$$

and in perturbative case

$$\begin{gathered}g=A(y,\alpha,\gamma)=\alpha-a_{20}\alpha^2-a_{11}\alpha\gamma-a_{02}\gamma^2+(2a_{20}^2+a_{11}c_{02}-a_{30})\alpha^3+\\ +(3a_{20}a_{11}+a_{11}c_{11}+2a_{02}c_{02}-a_{21})\alpha^2\gamma+(a_{11}c_{20}+a_{11}^2+2a_{20}a_{02}-a_{12}+\\ +2a_{02}c_{11})\alpha\gamma^2+(2a_{02}c_{20}-a_{03}+a_{11}a_{02})\gamma^3+\dots\end{gathered} \tag{5.51}$$

Analogous expansion for Γ can be obtained from here by subtitution $\alpha\to\gamma$ and $a_{ik}\to c_{ik}$.

Let us write down group differential Eqs. for G and H in two forms:

$$\frac{\partial G(x,y;g,h)}{\partial\ln x}=B(\frac{y}{x},G,H)\ ,\quad \frac{\partial H(x,y;g,h)}{\partial\ln x}=D(\frac{y}{x},G,H) \tag{5.52}$$

and

$$\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\beta(y,\alpha,\gamma)\frac{\partial}{\partial\alpha}-\delta(y,g,h)\frac{\partial}{\partial\gamma}\right]_{H}^{G}=0. \tag{5.53}$$

Generators B, D and β, δ are connected by the relations :

$${}^{\beta}_{\delta}(y,g,h)=\left[B(y,g,h)\frac{\partial}{\partial\alpha}+D(y,g,h)\frac{\partial}{\partial\gamma}-y\frac{\partial}{\partial y}\right]_{\Gamma}^{A}(y,\alpha,\gamma) \tag{5.54}$$

or

$${}^{B}_{D}(y,\alpha,\gamma)=[\beta(y,g,h)\frac{\partial}{\partial g}+\delta(y,g,h)\frac{\partial}{\partial h}-y\frac{\partial}{\partial y}]^{\alpha}_{\gamma}(y,g,h), \tag{5.55}$$

where quantities α, γ and g, h are related by Eqs. (19),(21). Given relations are analogs of (xx) and (yy). Note here that if, following the recipe (zz) we construct effective couplings for g and h :

$${}^{\bar{g}}_{\bar{h}}(x,y;g,h)={}^{A}_{\Gamma}\left(\frac{y}{x},G(x,y;g,h),H(x,y;g,h)\right), \tag{5.56}$$

then functions β and γ turn out to play the role of generators:

$$x\frac{\partial\bar{g}}{\partial x}=\beta(\frac{y}{x},g,h)\ ,\qquad x\frac{\partial\bar{h}}{\partial x}=\delta(y/x,g,h). \tag{5.57}$$

On the other hand if we introduce effective couplings for "new' coupling constants α and γ

$${}^{\bar{\alpha}}_{\bar{\gamma}}(x,y;\alpha,\gamma)={}^{G}_{H}\ (x,y,A(y/x,\alpha,\gamma),\Gamma(y/x,\alpha,\gamma)) \tag{5.58}$$

then B and D will appear as group generators:

$$x\frac{\partial\bar{\alpha}}{\partial x}=B(y/x,\bar{\alpha},\bar{\gamma})\ ,\quad x\frac{\partial\bar{\gamma}}{\partial x}=D(\frac{y}{x},\bar{\alpha},\bar{\gamma}) \tag{5.59}$$

It is worth mentioning that the relations (24,25) can be considered as describing transition (18) from coupling constants g, h and effective couplings G, H to 'new' ones α, γ and A, Γ , i.e. transition corresponding to the change of renormalization scheme, quite independent of the relation between invariant and effective couplings inside the given scheme.

4 RG Technique vs UV Divergencies

4.1 *RG functions and divergencies*

Renormgroup calculations in QFT often are performed on the basis of UV renormalization analysis. Due to this the very existence of quantum field RG is usually associated with presence of *UV* divergences. Meanwhile, as it has been shown above, *RG* can naturally be formulated without any reference to infinities. Here, we shall analyse the correspondence between *RG* in QFT and UV divergencies.

The most popular QFT problem solved with the help of RG-'mprovement' of perturbation theory results in the UV region by effective summation of infinite sets of logarithmic contributions. These contributions, as a rule, 'are born' by the logarithmically divergent expressions corresponding to 'logarithmic diagrams'. Such expressions after regularisations, but before renormalisation, contain logarithms of the ratio $Q^2/\Lambda^2, \Lambda$ being the regularisation parameter (e.g., cut-off momentum or Pauli-Villars mass) and Q – 4-momentum variable. For the renormalisation convenience such logs can be presented in the form

$$\ln(Q^2/\Lambda^2) = -\ln(\Lambda^2/\mu^2) + \ln(Q^2/\mu^2) \tag{5.60}$$

where μ - some arbitrary parameter with dimension of mass. The first term $L = \ln(\Lambda^2/\mu^2)$ is canceled by a counter-term, while the second $l = \ln(Q^2/\mu^2)$ 'goes' into renormalised finite expression for a given diagram with μ standing for momentum of subtraction or normalisation. The invariance with respect to change of μ value forms the basement of renormalisation group. Note also that in the case of dimensional regularisation the r.h.s. of Eq.(5.60) is equal to $1/\epsilon + l$

For our discussion the existence of a simple correspondence between asymptotic UV log $= l$ and divergent subtracted structure L (or $1/\epsilon$) is essential, i.e., between the contribution in the counter-term $\Delta\mathcal{L} \sim \mathcal{L}$ and UV asymptotic behaviour of some renormalized function $f_{ren}(Q^2) \sim \ell$.

For instance, the first radiative correction to the transverse amplitude of the QED photon propagator $\{= (\alpha/3\pi)\ell\}$ simply corresponds to the one-loop contribution $\{= (\alpha_o/3\pi)L\}$ to the renormalization constant Z_3. Correspondingly, the QED running coupling

$$\bar{\alpha}(Q^2) = \alpha(1 + \frac{\alpha}{3\pi}l) + O(\alpha^3) \tag{5.61}$$

and bare coupling α_0 considered as a function of L and α

$$\alpha_0 = \alpha Z_\alpha = \alpha\{1 + (\alpha/3\pi)L + O(\alpha^2)\} \tag{5.62}$$

can be expressed in terms of renormalised 'physical' coupling $\alpha = \bar{\alpha}(\mu^2)$ by similar expansions with L replaced by ℓ. Comparison of (5.61) with (5.62) reveals that, formally, the bare coupling α_0 from the initial Lagrangian can be considered as an UV limit of a running one.

We also see that the RG generator $\beta(\alpha)$ can be equally defined by any of two relations

$$\beta(\alpha) = \frac{\partial\bar{\alpha}}{\partial\ell}\Big|_{\ell=0} \tag{5.63}$$

or

$$\beta(\alpha) = \frac{\partial\bar{\alpha}_0}{\partial L}\Big|_{L=0} \tag{5.64}$$

Note also that in the case of dimensional regularisation an analog of (5.62) has the form

$$\alpha_B = \mu^{4\epsilon}\alpha\left(1+\sum_{\nu\geq 1}\frac{c_\nu(\alpha)}{\epsilon^\nu}\right) \tag{5.65}$$

and definition of beta-function is a bit more complicated :

$$\beta(\alpha) = \alpha^2 c_1'(\alpha) \tag{5.66}$$

As, from practical point of view, computation of divergent structures turns out to be more simple than calculation of momentum asymptotics, especially for vertex functions, definitions (5.64) and (5.66) became rather popular.

It is worth noticing that in this case β-function defined through (product of) renormalisation constant(s) is used then for solution of RG differential Eqs. for finite renormalised functions.

4.2 *Mass dependence*

Now let us go on to the case when masses of field quants are essential. Here, instead of eq.(5.60), we have

$$I_{reg}(Q^2, m^2, \Lambda^2) = I_{sing}(\Lambda, m, \mu) + I_{ren}(Q^2, m^2, \mu^2) \tag{5.67}$$

The way of singular part separating in the r.h.s. is defined by renormalisation prescription, that is by renormalisation scheme (RS). For instance, in the case of momentum subtraction (MOM-scheme) for logarithmic divergence we have

$$I_{ren}(Q^2, m^2, \mu^2) = \lim_{\Lambda\to\infty}\{I_{reg}(Q^2, m^2, \Lambda) - I_{reg}(\mu^2, m^2, \Lambda)\} = I(Q^2/m^2) - I(\mu^2/m^2), \tag{5.68}$$

$$I_{\text{sing}}(\Lambda, m, \mu) = I^{\text{reg}}(\mu^2, m^2, \Lambda)$$

and in the limit $\Lambda \to \infty$

$$I^{reg}(\mu^2, m^2, \Lambda) - \to -\ln(\Lambda^2/\mu^2) + I(\mu^2/m^2) + \text{ const.} \tag{5.69}$$

Chapter VI

RENORMGROUP in QFT. Details

1 Reliability of UV results in QFT

1.1 *Ghost pole in QED*

It must be noted that in perturbation calculation, the β function depends on the adopted renormalization procedure; in the 3-loop or higher level the coefficients of the perturbation series depend on Renormalization Scheme(RS) used. In QED, the three loop β function in MOM (i.e. momentum subtraction) scheme is expressed as

$$\beta^{MOM}_{3-loop} = \frac{\alpha^2}{3\pi} + \frac{\alpha^3}{4\pi^2} + \frac{\alpha^4}{8\pi^3}\left(\frac{8}{3}\zeta(3) - \frac{101}{36}\right) \quad .$$

The numerical value of the last parenthesis is about 0.4 . Neglecting it for the moment we start our discussion with 2-loop approximation for the β function. According to Eq.(II-29) the 2-loop RG solution is

$$\bar{\alpha}_{(2)} = \frac{\alpha}{1 - \frac{\alpha}{3\pi}l + \frac{3\alpha}{4\pi}\ln(1 - \frac{\alpha}{3\pi}l)} \quad .$$

This solution has an error of order $\alpha^4 l$ and is interesting from several points of view. As it was mentioned before, its α expansion besides leading logs contains an infinite number of next-to-leading terms $\alpha^2(\alpha l)^m$, the first of which has been used as an input for β function construction. Second, in the vicinity of the ghost pole of one-loop RG solution at $l_1 = 3\pi/\alpha$, the $\bar{\alpha}_{(2)}$ solution differs from $\bar{\alpha}_{(1)}$ considerably. Hence, the infinite sum of the next-to-leading logarithmic contributions in the region $\alpha l \sim 1$ becomes important. It is not trivial because inside the each order of the perturbative input the next-to-leading term is negligible comparing with the leading one (the ratio being of order $\alpha\beta_2/\beta_1 = 3\alpha/4\pi \approx 2.10^{-3}$). It can be shown that the accounting the last, 3-loop, term in eq.(5.8) also becomes essential for the $\bar{\alpha} \sim 1$ case.

This means that the existence problem for the ghost pole in QED can not be solved by taking into the account next-to-leading and so on logs. Moreover, one can conclude that it is impossible to make any qualitative statement about UV asymptote for $\beta(g) > g$ case basing on a RG-improved perturbation calculations. Our next example illustrates this thesis.

1.2 *Ghost pole in QCD - absent*

1.3 *Scalar quartic model*

Consider the nonlinear scalar field with the interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{4\pi^2}{3} g\phi^4 \quad .$$

Here, considerable progress has been achieved in the calculation in the higher perturbative diagrams. In particular, the β function in MS-scheme is known now up to 5-loop level [29], [30] and the result is given by

$$\beta^{MS}_{(5)} = \frac{3}{2}g^2 - \frac{17}{6}g^3 + 16.275g^4 - 135.8g^5 + 1437g^6 \quad . \tag{6.1}$$

The number of diagrams in each perturbation order are 1, 2, 7, 23 and ~ 100, respectively. The shape of the β function up to each loop order is shown in Fig. VI.1.

Figure VI.1: Betha-function in the φ^4 model.

We see from this nice picture that, in conrast with QED case, there is no stability here even on a qualitative level. The odd-order approximations (i.e.,1-, 3- and 5-loop) have a ghost-pole type behavior, whereas the even ones yield the fixed point (finite charge renormalization) case. Note also that, as shown at Fig. VI.2, the 10% confidence region upper boundaries correspond to g values close to 0.1 . To understand the "loop dependence" of this boundary it is useful to represent the expression (6.1) in a slightly different form

$$\beta^{MS}(g) = \frac{3}{2}g^2\left[1 - \frac{g}{0.529} + \left(\frac{g}{0.303}\right)^2 - \left(\frac{g}{0.222}\right)^3 + \left(\frac{g}{0.180}\right)^4\right] \quad .$$

This expression looks like a beginning of an asymptotic series of Poincaré type. Indeed if we represent the β-function in a series expansion form

$$\beta(g) = \sum_n g^{n+1}\beta_n \quad , \tag{6.2}$$

then, as it is well known, coefficients β_n behave like $\sim n!$ at $n \to \infty$.

The method of determining asymptotic estimates of the perturbation expansion coefficients of the Green functions uses a representation in the form of a functional (i.e. path) integral. This integral written down for the mentioned expansion coefficient can be calculated by the steepest descent method in the function space first proposed in Ref. [31]. To the saddle point there corresponds the "instanton" type Euclidean classical solution with finite action. In this manner, an asymptotic expression was obtained by Lipatov [32] for the coefficients of β function expansion in eq. 6.2. It has the form

$$\beta_n \simeq \frac{1.096}{16\pi^2} n! n^{7/2} \left(1 + O(n^{-1})\right) \qquad (n \to \infty) \ ,$$

The factorial coefficients growth indicates that this is an asymptotic series with zero radius of convergence and that it cannot be summed in the usual manner. We can obtain the information about singularity structure at the origin ($g = 0$) by using some special procedures. One of them is the Borel summation method.

1.4 *Conclusion - absent*

2 Asymptotic series Borel summation

Here we give short exposition of the resuts on the attempt of the summation of the series of (6.1) type made in [33] (see also review paper [34]).

Figure VI.2: Confidence region ... .

Authors of the Ref. [33] used as an input the β-function 4-loop expression in the symmetric MOM-scheme

$$\beta_5^{MOM}(g) = \frac{3}{2}g^2 - \frac{17}{6}g^3 + 19.3g^4 - 146g^5 \ .$$

The alternate-sign asymptotic series can be summed by Borel method. The idea is to represent the sum in a form of Laplace transform integral. It is not difficult to see that the transition to the Laplace image "kills" the factorial factor $n!$. For the modified Borel transformation defined as

$$\beta(g) = \int_0^\infty \frac{dx}{g} \exp\left(-x/g\right) \left(x \frac{\partial}{\partial x}\right)^5 B(x) \ . \tag{6.3}$$

perturbation series can be written down as

$$B(x) = \sum_n \frac{\beta_n}{n! n^5} x^n \quad ,$$

Now it has a nonzero circle of convergence and can be summed within the circle. However, as the integration domain in eq. 6.3 goes outside the convergency region, we must make an analytic continuation for the function $B(x)$. It can be done by a conformal mapping of the x-plane into the w-plane in such a way that the domain of integration $[0, \infty]$ is mapped into the interior of the unit disk and the cut $[-\infty, -1]$ transforms into the boundary of the disk. We also choose the conformal transformation in such a way that it correctly reproduces the singularity of the function on the cut. A conformal mapping satisfying these conditions is

$$w(x) = \frac{\sqrt{1+x} - 1}{\sqrt{1+x} + 1} \quad .$$

Then the Borel image of (5.26) becomes

$$B(x) = \frac{3x^2}{128}(1 - 0.32w - 0.127w^2 + 0.084w^3) \ , \tag{6.4}$$

at the 4-loop approximation. Analyticity properties in x- and w-plane are explained in Fig. VI.3.

Graphs of the function $\beta(g)$ obtained by the Borel summation with allowance for the 1-, 2-, 3- and 4-loop approximations are shown in Fig. VI.4. They all lie in a narrow ray below the original parabola and within the limit of our adopted 10% accuracy enables us to advance into the region $g \sim 50$. This means that the summation procedure adopted enlarges the confidence interval in several hundred times! Besides this it give us the qualitative stability of results. All they are in a favour of ghost-type UV asymptote.

Nethertheless the results obtained can be considered only as a support but not the proof of the ϕ^4-model inconsistency. The point is that starting with the representation (27) we have assumed the definite analyticity properties of $\beta(g)$ in the whole complex g-plane.

3 Gauge dependence in QCD

Here we give a brief description of the resuls obtained in papers [28],[29].

3.1 *Running of a coupling and of a gauge*

Consider QCD in a relativistic invariant gauge when the gauge-fixing term in the Lagrangian is of the form

$$\Delta\mathcal{L} = \frac{1}{2a}(\partial_\mu B^\mu)^2 \quad ,$$

with constant gauge parameter a introduced. It is known since the mid-fifties [30] that in the RG analysis one has to treat the gauge parameter as the second coupling constant. Then we should define the *effective gauge* coupling

$$a \to \bar{a}(Q) = ad^{-1}\left(\frac{Q^2}{\mu^2}; \alpha, a\right) \quad ,$$

with the transverse gluon propagator amplitude

$$q^2 D_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) d\left(\frac{Q^2}{\mu^2}\right) + \frac{q_\mu q_\nu}{q^2} a$$

involved. This means that if we use constant non-zero gauge $a \neq 0$ then we are in a two-coupling situation. By setting $\alpha \equiv \alpha_s = g^2/4\pi$ the RG equations become(see, e.g. [29].

$$\frac{\partial\bar{\alpha}(l;\alpha,a)}{\partial l} = \beta(\bar{\alpha},\bar{a}) \ , \qquad \frac{\partial\bar{a}(l;\alpha,a)}{\partial l} = b(\bar{\alpha},\bar{a}) \quad .$$

In MS scheme, the β function does not depend on a, however, it depends on a in MOM scheme.

Figure VI.3: x complex plane vs w-plane.

3.2 Gauge phantoms in QCD

It is possible to have a non-trivial fixed point (α_*, a_*) in $(\bar{\alpha}, \bar{a})$ plane:

$$\beta(\alpha_*, a_*) = 0 \quad , \quad b(\alpha_*, a_*) = 0 \ .$$

Such situation does have place in two-loop approximation for some popular MOM-schemes. Some examples of trajectories, which occur at two-loop level, in $(\bar{\alpha}, \bar{a})$ plane are shown in Fig. VI.5. If we start with some negative gauge parameter low-energy value $a < 0$, we end in a fixed point (α_*, a_*). In this solution the asymptotic freedom is not accomplished ($\alpha_* \neq 0$). On the other hand, trajectories starting from positive gauge parameter values show asymptotic free (in $\bar{\alpha}_s$) behavior and converge to a fixed gauge parameter value. The trajectorie in $(\bar{\alpha}, l)$ plane are also shown in Fig. VI.6. The main lesson from this exercise is that the effective coupling in QCD shows both scheme dependence and gauge dependence so it cannot be a physical observable.

3.3 *Stopping the gauge running* – *absent*

4 Massive RG in QCD - d'inachevé

4.1 *Formulation of the Problem*

As far as quark masses are quantities of different orders of magnitude, in QCD we meet a specific situation of "mass account in UV asymptotic" expressions. The point is that there exists a rather wide physically important momentum transfer $Q = \sqrt{Q^2}$ interval that can be considered as asymptotic with respect to light quark masses on one hand but containing heavy quark mass thresholds on the other.

To formulate the problem consider the case when we have a light particle (group of particles) with small mass(es) m and heavy one(s) with big mass M such that

$$\ln(M^2/m^2) >> 1 \tag{6.5}$$

Figure VI.4: Borel-improved β-function for φ^4 model.

Figure VI.5: Phase portrait ... .

Figure VI.6: Fig[VI.12]

Then the whole Q variable interval can be divided on three parts: Low asymptotic region

$$\ln(Q^2/m^2) >> 1 \ , \quad Q^2 << M^2; \tag{6.6}$$

Intermediate (threshold) region

$$Q^2 \tilde{} M^2; \qquad (2b)$$

High asymptotic region

$$\ln(Q^2/M^2) >> 1. \qquad (2c)$$

In the first of them the contributions of diagrams with light massive internal lines are in the logarithmic regime, i.e., $\sim \ln(Q^2/m^2)$, whereas the contributions of heavy massive diagrams due to decoupling property are negligibly small. These heavy mass contributions become important in the intermediate region where they have specific threshold behaviour and acquire an asymptotic form [$\sim \ln(Q^2/M^2)$] in the high asymptotic region.

To illustrate, consider the QCD gluonic propagator. Its dimensionless transverse amplitude in the one-loop approximation has the form

$$d(Q^2, \ldots) = 1 - \frac{\alpha_s}{4\pi} J(Q^2, ...) \tag{6.7}$$

with

$$J(Q^2, \ldots) = 11 \ln(Q^2/m^2) - \frac{2}{3} \sum_q \{I(Q^2/m_q^2) - I(m^2/m_q^2)\}, \tag{6.8}$$

where I(z) is the fermion-loop contribution

$$I(z) = 6 \int_0^1 dt(1-t)t \ln[1 + 4zt(1-t)] \tag{6.9}$$

with properties

$$I(0) = 0, I(z) \to \ln z - 0.28 \ \text{ as } \quad z \to \infty.$$

The first term in the r.h.s. of eq.(6.8) represents the gluon contribution while the second that of different quarks. The transition from "low" to "high" asymptotic regime corresponds here to transition from $f = 3$ to $f = 5$ region, f being the number of quark flavours.

4.2 *Massive 1-loop solution - absent*

4.3 *Massive 2-loop solution - absent*

5 RG in Standard Model - d'inachevé

5.1 *Preamble - absent*

5.2 *Massless 2-loop coupling evolution - absent*

On the other hand at the same time this approach was developed further in our paper [35], where it was shown that its the most remarkable feature[1] that *RG-summed results can be expressed in terms of usual perturbation coefficient functions* can be supported, at least approximately, in few higher orders of RG approximations.

5.3 *Massive 2-loop coupling evolution*

The technical means of our discussion is an analytic solution of the system of RG evolution equations for three effective couplings of the Standard Model in a mass-dependent case. To obtain it, we start with a two-loop perturbative input for three effective couplings of the Standard Model

$$\alpha_i(Q) \simeq \alpha_i - \alpha_i^2 A_i(Q) + \alpha_i^3 A_i^2(Q) - \alpha_i^2 \sum_j \alpha_j A_{ij}(Q) \tag{6.10}$$

where A_i and A_{ij} are one- and two-loop mass- and μ-dependent contributions of appropriate Feynman diagrams. In the framework of a *massive renorm-group formalism* [2] the corresponding Lie differential equations look like

$$\dot{\alpha}_i(Q) \simeq -\alpha_i^2(Q)\left[\dot{A}_i(Q) + \sum_j \alpha_j(Q)\dot{A}_{ij}(Q)\right] \tag{6.11}$$

with $\dot{A} \equiv Q\partial A/\partial Q;\ Q = \sqrt{Q^2}$. Note that in the UV, i.e.- pure massless, limit

$$A_i(Q) = \beta_i l\ ;\quad A_{ij}(Q) = \beta_{ij} l\ ;\quad l = \ln Q/\mu \tag{6.12}$$

we arrive at the system

$$\dot{\alpha}_i(Q) = -\left(\beta_i + \sum_j \beta_{ij}\alpha_j(Q)\right)\alpha_i^2(Q)$$

that is commonly used

for discussion of the extrapolation of the experimental data across gauge desert and possibility of Grand Unification.

[1]See below Eqs.(6.14),(6.15).

As it is well known the latter system can be solved iteratively in the form

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i} + \beta_i l + \sum_j \frac{\beta_{ij}}{\beta_j} \ln[1 + \alpha_j \beta_j l] \tag{6.13}$$

with $\alpha_i = \alpha_i(\mu)$.

In this paper we present a generalization of this solution to the massive case that is convenient for taking account threshold effects and discussing the issue of Grand Unification consistency check. Using the method of the paper[35] we can obtain an explicit iterative solution to the system (6.11). Here, as in the massless case (6.12),(6.13), one has to solve first the one-loop approximation to (6.11) and obtain[2]

$$\alpha_i^{(1)}(Q) = [1/\alpha_i + A_i(Q)]^{-1} \ . \tag{6.14}$$

Insert now the explicit expression $\alpha_j^{(1)}(Q)$ into the second factor in the r.h.s. of Eq.(6.11). After this one can separate variables and has to consider integrals

$$\int^Q \frac{\dot{A}_{ij}(q)dq/q}{1 + \alpha_j A_j(q)} \ .$$

Technique of an approximate integration of such expressions was devised in the mentioned paper [35]. Using it we arrive at the expression

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i} + A_i(Q) + \sum_j \frac{A_{ij}(Q)}{A_j(Q)} \ln[1 + \alpha_j A_j(Q)] \ . \tag{6.15}$$

The remarkable feature of this solution is that it depends explicitely only on mass-dependent perturbation coefficients $A_i(Q)$, $A_{ij}(Q)$ and, being expanded in powers of coupling constants, exactly corresponds to the perturbative input (6.10). On the other hand, in the massless limit it goes to the solution (6.13).

The accuracy of expression (6.15) can be estimated by the method used in the paper[37]. Generally, it corresponds to the accuracy of a three-loop expression[3] for the effective coupling in the one-coupling case and is quite enough for our discussion.

[2] This exact solution of a massive RG equation was first obtained in mid-fifties – see Ref.[36].
[3] See Eqs.(13),(16) in Ref.[37].

Part 4: Bogoliubov RG

Chapter VII

FUNCTIONAL SELF-SIMILARITY in MATHEMATICAL PHYSICS

This chapter is devoted to transmission of RG ideas born at QFT and widely used in the theory of large statistical systems to more simple domain of classical mathematical physics.

1 FSS and Lie group analysis

The simplest variant of RG transformations is described by a functional equation for a solution of some physical problem $\bar{g}(x,g)$, that depends upon two arguments,[1] – one dimensionless "coordinate" x and one solution parameter g

$$\bar{g}(x,g) = \bar{g}(x/a, \bar{g}(a,g)), \quad \bar{g}(1,g) = g \ , \tag{7.1}$$

or a corresponding one-parameter group of point transformations

$$x' = x/a \ , \quad g' = \bar{g}(a,g) \ , \quad x'|_{a=1} = x \ , \quad g'|_{a=1} = g \ . \tag{7.2}$$

The functional equation (7.1) and RG transformation (7.2) arise, for example, in the massless quantum field theory with one coupling. In that case $\bar{g}$ is the effective coupling, $x = Q^2/\mu^2$ is the ratio of a 4-momentum Q squared to a "reference momentum" μ squared and g is the coupling constant. [2].

The use of transformations (7.2) in (7.1) in the limit $a \to 1$ gives the differential representation of the RG functional equation

$$R\bar{g} = 0 \, , \tag{7.3}$$

[1]The generalization to the case of several solution parameters is strightforward .

[2]In the modern notations [8].

where R is the *infinitesimal operator* of RG-symmetry (or simply RG-operator) with the coordinate β defined by the first derivative of the function $\bar{g}$ at the point $a = 1$

$$R = x\partial_x - \beta(g)\partial_g \,, \quad \beta(g) = \left.\frac{\partial \bar{g}(a;g)}{\partial a}\right|_{a=1} . \tag{7.4}$$

Therefore, instead of the functional equation (7.1), RG transformation can be introduced by means of an RG-operator, as well. And vice versa, being given an RG-operator, one can easily reconstruct the functional equation for the solution $\bar{g}$ with the help of the corresponding Lie equations

$$\begin{gathered} \frac{dx'}{x'} = -\frac{da}{a}, \quad \frac{dg'}{\beta(g')} = \frac{da}{a}, \quad d\bar{g}' = 0; \\ x'|_{a=1} = x, \; g'|_{a=1} = g, \; \bar{g}'|_{a=1} = \bar{g}. \end{gathered} \tag{7.5}$$

Thus, for given values of the β-function defined by the behavior of $\bar{g}$ in the vicinity of $a = 1$ or, in other words, for given RG-symmetries, one can get the solution of a physical problem under consideration. However, the procedure of finding RG-symmetries in any specific case up to now usually leads to atypical manipulations [56, 57] and the regular approach to revealing RG-symmetries is of principal interest.

2 Approach to constructing RG-symmetries

The desired regular approach to constructing RG-symmetries appears to be possible for those mathematical models of physical systems that are based on differential equations. The key idea for this constructing lies in the fact that these models can be analyzed using algorithms of modern group analysis.

The approach [58] suggested to obtain RG-symmetries comprises a sequence of the following steps[3]. A specific manifold (differential, integro-differential, etc.) should be primarily constructed, which is then used to calculate RG-symmetries. This manifold, that will be referred to as *renormgroup manifold* (RG-manifold), generally differs from the manifold given by the original system of differential equations (DEs). The second step deals with calculating the most general group, which is admitted by RG-manifold. The restriction of the group obtained on the solution (exact or approximate) of the b.v.p. constitutes the next, third step in the procedure of constructing RG-symmetries. The group of transformations thus obtained (renormgroup) is characterized by a set of infinitesimal operators (RG-operators), each containing the solution of a b.v.p. in its invariant manifold. The last, fourth step, implies utilization of RG-operators to find analytical expressions for solutions of the b.v.p..

[3]It should noticed, however, that the suggested approach to RG constructing is new and differs from the traditionally used in theoretical physics. As a simple illustration of this difference we point on an extended interpretation of the notion of the renormalization group.

In the scheme described above the first step, namely constructing RG-manifold, is of fundamental importance. It depends both on a mathematical model used for the analyzed physical system and on a form in which boundary conditions are taken into account. Different approaches to RG-manifold constructing lead to the following different methods of finding RG-symmetries. In the first, most simple, case the RG-manifold as usually in classical group analysis is presented by the system of basic DE with the only substantial difference: parameters, entering a solution via equation and boundary data, are included in the list of independent variables. The second approach to constructing RG-manifold implies the extension of a space of variables involved in group transformations for example, by taking into account Lie-Bäcklund transformation groups and nonlocal transformation groups. The third approach is based on combining the group analysis technique with the invariant embedding method [59]. In this case group analysis is fulfilled for the system of equations that consists of original DE and/or embedding equations which correspond to the b.v.p. under consideration. The fourth approach is to some extend similar to that based on embedding equations. In this event boundary conditions are reformulated in terms of differential constraint which is combined with original equations to give the desired RG-manifold. The last, fifth approach to constructing RG-manifold utilizes approximate transformation groups. Here RG-manifold is given by a system of DEs with small parameters and can be analyzed by perturbation methods.

Whereas the second step is a standard well-defined procedure in group analysis one further comment should be made concerning the third step, namely the procedure of restriction of the group obtained on the solution (exact or approximate) of the b.v.p.. The goal of such restriction is the construction of transformation group with tangent vector field (point, Lie-Bäcklund, etc.) infinitesimal operators of which contain the desired solution of the boundary value problem in invariant manifold. That means, that coordinate of the canonical operator of RG-symmetry becomes equal to zero on the solution of the boundary value problem and its differential consequences.

Mathematically the procedure of group restriction appears as "combining" different coordinates of group generators admitted by RG-manifold. The zero-equality condition for the sum of these coordinates on a solution of the b.v.p. leads to algebraic equalities that couple different coordinates and give rise to desired RG-symmetries. In a particular case when RG is constructed from the Lie group, admitted by the original system of DE, it appears as a subgroup of this group and a solution of the b.v.p. results as invariant solution with respect to the point RG obtained (compare with [60]). In the general case not only Lie point group, but Lie-Bäcklund groups, approximate groups, nonlocal transformation groups, etc., are also employed as basic groups, which are then restricted on the solution of a b.v.p..

The final step, i.e. constructing analytical solutions of the b.v.p. on the basis of RG-symmetries obtained, needs no comments.

The merits of the described approach were first realized while solving the problem of nonlinear interaction of a powerful laser radiation with inhomogeneous plasma [61]. The mathematical model of a process is given by a system of nonlinear differential equations for components of electron velocity, electron density and the electric and the magnetic fields. The presence of small parameters in the original system of equations (such as weak inhomogeneity of ion density, low electron thermal pressure and small angles of incidence of the laser beam on plasma) enables to construct RG-symmetries using the approximate group methods. In a particular case that is of interest from the physical standpoint the desired RG-symmetries appears as a Lie point symmetry that takes into account transformations of the nonlinearity parameter which is proportional to the amplitude of the magnetic field at the laser frequency at the critical density point and is assumed to be small in the perturbation theory approximation. RG-symmetries obtained enable to reconstruct the exact solution of original equations from the perturbation theory results. The solutions found were used to calculate the nonlinear structure of the electromagnetic field in plasma and to evaluate the efficiency of harmonics generation.

Further development of the approach was concerned with the initial value problem for modified Burgers equation with explicitly included parameters of nonlinearity and dissipation. This example [62] yielded a detailed illustration of the method of constructing RG-symmetries when a basic RG-manifold is given by an original differential equation with parameters included in the list of independent variables.

To demonstrate the method of constructing Lie-Bäcklund RG-symmetries the initial value problem for a linear parabolic equation was studied in [58]. The same mathematical model was also employed to find RG-symmetries when the boundary condition is described by a differential constraint [64].

An idea of a specific method of constructing RG-symmetries which is based on embedding equations was first treated in [65] for ordinary differential equations.

Worthy of mention is an example which demonstrates utilization of RG-symmetries to constructing solutions of the b.v.p. for a system of two first order partial differential equations, that describes a propagation of a laser beam in nonlinear focusing medium [66] Both Lie point and Lie-Bäcklund symmetries (exact and approximate) were used to find RG-symmetries, which were then utilized to find analytical solution of the problem.

To clarify the idea of constructing RG-symmetries several examples are given below, which demonstrate different approaches to the problem. To gain better understanding of these approaches a unique simple mathematical model is used[4] that corresponds to b.v.p for a system of two first order partial differential equations that were first treated by Chaplygin [67] in gas dynamics

$$\begin{gathered} v_t + vv_x - a\varphi(n)n_x = 0\,, \quad n_t + vn_x + nv_x = 0\,; \\ v(0,x) = V(x)\,, \quad n(0,x) = N(x)\,, \end{gathered} \tag{7.1}$$

where $\varphi(n)$ is an arbitrary function of n and a is a nonlinearity parameter. Despite its simplicity this model has a wide field of application and is used to describe various phenomena It should be noted that for different problems the physical meaning of variables t, x, v and n may differ from that in gas dynamics. For example, in nonlinear

[4] A disadvantage of this is that some examples have a methodological, rather than a physical significance.

geometrical optics t and x are coordinates respectively along and transverse to direction of laser beam propagation, v is derivative of eikonal with respect to x, and n is a laser beam intensity. In this case functions V and N characterize the curvature of the wave front and the beam intensity distribution upon the coordinate x and the entrance of a medium $t = 0$.

Appendix A

HISTORY OF THE RG IN QFT

A1 Introduction

A1.1 *Renormalization*

As is known, the regular formalism for eliminating ultraviolet divergences in quantum field theory (QFT) was developed on the basis of covariant perturbation theory in the late 40s. This breakthrough is connected with the names of Tomonaga, Feynman, Schwinger and some others. In particular, Dyson and Abdus Salam carried out the general analysis of the structure of divergences in arbitrarily high orders of perturbation theory. Nevertheless, a number of subtle questions concerning so-called overlapping divergences in the scattering matrix, as well as surface divergences, discovered by Stueckelberg[71] in the Tomonaga–Schwinger equation, remained unclear.

An important contribution in this direction based on a thorough analysis of the mathematical nature of UV divergences was made by Bogoliubov. This was achieved on the basis of a branch of mathematics which was new at that time, namely, the Sobolev–Schwartz *theory of distributions*. The point is that propagators in local QFT are distributions (similar to the Dirac delta–function) and their products appearing in the coefficients of the expansion of the scattering matrix require supplementary definitions. In view of this the UV divergences reflect the ambiguity in the definition of the products in the case when their arguments coincide or lie on the light cone.

In the mid 50s on the basis of this approach Bogoliubov and his disciples developed a technique of supplementing the definition of the products of singular Stueckelberg–Feynman propagators [72] and proved a theorem [73] on the finiteness and uniqueness (for renormalizable theories) of the scattering matrix in any order of perturbation theory. The prescription part of this theorem, namely, *Bogoliubov's R-operation*, still remains a practical means of obtaining finite and unique results in perturbative calculations in QFT.

The R–operation works, essentially, as follows. To remove the UV divergences, instead of introducing some regularization, for example, the momentum cutoff, and handling quasi-infinite counterterms, it suffices to complete the definition of divergent Feynman integrals by subtracting from them certain polynomials in the external momenta which in the simplest case are reduced to the first few terms of the Taylor series of the integral. The uniqueness of computational results is ensured by special conditions imposed on them. These conditions contain specific degrees of freedom[1] that can be used to establish the relationships between the Lagrangian parameters (masses, coupling constants) and the corresponding physical quantities. The fact that physical predictions are independent of the arbitrariness in the renormalization conditions, that is, they are *renormalization–invariant*, constitutes the conceptual foundation of the renormalization group.

An attractive feature of this approach is that it is free from any auxiliary nonphysical attributes such as bare masses, coupling constants, and regularization parameters which turn out to be unnecessary in computations employing Bogoliubov's approach. As a whole, this method can be regarded as *renormalization without regularization and counterterms.*

A1.2 *The discovery of the renormalization group*

The renormalization group approach has been known in theoretical physics since the mid 50s. The renormalization group was discovered by Stueckelberg and Petermann [1] in 1953 as a group of infinitesimal transformations related to the finite arbitrariness arising in the elements of the scattering S-matrix upon elimination of the ultraviolet divergences. This arbitrariness can be fixed by means of certain parameters c_i:

> *"... we must expect that a group of infinitesimal operators* $\mathbf{P}_i = (\partial/\partial c_i)_{c=0}$, *exists, satisfying*
> $$\mathbf{P}_i S = h_i(m,e)\partial S(m,e,...)/\partial e \ ,$$
> *admitting thus a renormalization of e."*

These authors introduced the *normalization group* generated (as a Lie group) by the infinitesimal operators $\mathbf{P}_i$ connected with the renormalization of the coupling constant e.

In the following year, on the basis of Dyson's transformations written in the regularized form, Gell-Mann and Low [6] derived functional equations for QED propagators in the UV limit. For example, for the renormalized transverse part d of the photon propagator they obtained an equation of the form

$$d\left(\frac{k^2}{\lambda^2}, e_2^2\right) = \frac{d_C(k^2/m^2, e_1^2)}{d_C(\lambda^2/m^2, e_1^2)} \ , \quad e_2^2 = e_1^2 d_C(\lambda^2/m^2, e_1^2) \ , \tag{1.1}$$

[1] These degrees of freedom correspond to different renormalization schemes and momentum scales.

where λ is the cutoff momentum and e_2 is the physical electron charge. The appendix to this article contains the general solution (obtained by T.D.Lee) of this functional equation for the photon amplitude $d(x, e^2)$ written in two equivalent forms:

$$e^2 d\left(x, e^2\right) = F\left(xF^{-1}\left(e^2\right)\right)$$

and

$$\ln x = \int_{e^2}^{e^2 d} \frac{\mathrm{d}y}{\psi(y)} \,, \tag{1.2}$$

where

$$\psi(e^2) = \frac{\partial(e^2 d)}{\partial \ln x} \quad \text{at} \quad x = 1 \,.$$

A qualitative analysis of the behaviour of the electromagnetic interaction at small distances was carried out with the aid of (1.2). Two possibilities, namely, infinite and finite charge renormalizations were pointed out:

> *Our conclusion is that the* **shape** *of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through the scale factor. The behavior of the propagator functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e_1^2/4\pi\hbar c$ with divergent coefficients, many behave either in two ways:*
>
> *It may really be infinite as perturbation theory indicates;*
>
> *It may be a finite number independent of $e_1^2/4\pi\hbar c$.*

The latter possibility corresponds to the case when ψ vanishes at a finite point:[2] $\psi(\alpha_\infty) = 0$.

We remark that paper [6] neither paid attention to the group character of the analysis and the results obtained, nor paper [1] quoted. Moreover, the authors did not recognize that the Dyson transformations used by them are valid only for the transverse scaling of the electromagnetic field. Maybe this is why they failed to establish a connection between their results and the standard perturbation theory and they did not discuss the possibility that a ghost pole might exist.

The final step was taken by Bogoliubov and Shirkov [2], [3] [3]. Using the group properties of finite Dyson transformations for the coupling constant and the fields, the

[2]Here α_∞ is the so–called fixed point of the renormalization group transformations.

[3]See also the survey [5] published in English in 1956.

authors obtained functional group equations for the propagators and vertices in QED in the general case (that is, with mass taken into account). For example, the equation for the transverse amplitude of the photon propagator was obtained in the form

$$d(x, y; e^2) = d(t, y; e^2)d\left(x/t, y/t; e^2 d(t, y; e^2)\right) \ ,$$

in which the dependence of d not only on $x = k^2/\mu^2$ (where μ is a certain normalizing scale factor), but also on the mass variable $y = m^2/\mu^2$ is taken into account.

In the modern notation, the above relation [4] is an equation for the square of the effective electromagnetic coupling constant $\bar{\alpha} = \alpha d(x, y; \alpha = e^2)$:

$$\bar{\alpha}(x, y; \alpha) = \bar{\alpha}\left(x/t, y/t; \bar{\alpha}(t, y; \alpha)\right) \ . \tag{1.3}$$

The term "*renormalization group*" was introduced and the notion of *invariant charge* [5] was defined in [2].

Let us emphasize that, in contrast to the Gell-Mann and Low approach, in our case there are no simplifications due to the massless nature of the ultraviolet asymptotics. Here, the homogeneity of the mass scale is violated explicitly by the scale term m. Nevertheless, the symmetry (even though a bit more complex one) underlying the renormalization group can, as before, be stated as an *exact symmetry* of the solutions of the quantum field problem.This is what we mean when using the term *Bogoliubov's renormalization group* or **Renorm-group** for short.

The following differential group equations for $\bar{\alpha}$:

$$\frac{\partial \bar{\alpha}(x, y; \alpha)}{\partial \ln x} = \beta\left(\frac{y}{x}, \bar{\alpha}(x, y; \alpha)\right) \tag{1.4}$$

in the nonlinear form, which is standard in Lie theory, and for the electron propagator $s(x, y; \alpha)$:

$$\frac{\partial s(x, y; \alpha)}{\partial \ln x} = \gamma\left(\frac{y}{x}, \bar{\alpha}(x, y; \alpha)\right) s(x, y; \alpha) \ , \tag{1.5}$$

where

$$\beta(y, \alpha) = \frac{\partial \bar{\alpha}(\xi, y; \alpha)}{\partial \xi} \ , \qquad \gamma(y, \alpha) = \frac{\partial s(\xi, y; \alpha)}{\partial \xi} \quad \text{at} \ \ \xi = 1 \ . \tag{1.6}$$

were first obtained by differentiating the functional equations. In this way an explicit realization of the differential equations mentioned in the citation from [1] was obtained. These results established a conceptual link between the Stueckelberg – Petermann and Gell-Mann – Low approaches.

A1.3 *Creation of the RG method*

Another important achievement of paper [2] consisted in formulating a simple algorithm for improving an approximate perturbative solution by combining it with the Lie

[4] In the massless case $y = 0$ it is equivalent to (1.4).

[5] This notion is now widely known as the effective or running coupling constant.

equations[6]:

> *Formulae (1.4) and (1.5) show that to obtain expressions for $\bar{\alpha}$ and s valid for all values of their arguments one has only to define $\bar{\alpha}(\xi, y, \alpha)$ and $s(\xi, y, \alpha)$ in the vicinity of $\xi = 1$. This can be done by means of the usual perturbation theory.*

In the next publication [3] this algorithm was used effectively to analyse the ultraviolet and infrared (IR) asymptotic behaviour in QED in transverse gauge. The one-loop and two-loop UV asymptotics

$$\bar{\alpha}^{(1)}_{RG}(x,0,\alpha) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\cdot\ln x} \tag{1.7}$$

and

$$\bar{\alpha}^{(2)}_{RG}(x,0,\alpha) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln x + \frac{3\alpha}{4\pi}\ln(1 - \frac{\alpha}{3\pi}\ln x)} \tag{1.8}$$

of the photon propagator as well as the IR asymptotics

$$s(x,y;\alpha) \approx (p^2/m^2 - 1)^{-3\alpha/2\pi} \ .$$

of the electron propagator were obtained. At that time these expressions had already been known only at the one–loop level. It should be noted that in the mid 50s the problem of the UV behaviour in local QFT was quite urgent. Substantial progress in the analysis of QED at small distances was made by Landau and his collaborators [74] by solving an approximate version of the Schwinger–Dyson equations including only the two-point functions ("dressed" propagators) $\Delta_i(..., \alpha)$ and the three-point function $\Gamma(..., \alpha)$, that is, the so-called "three-gamma equations". The authors obtained asymptotic expressions for QED propagators and 3-vertex, in which (using modern language) the leading UV logarithms were summed [7]. However, Landau's approach did not provide a prescription for constructing subsequent approximations.

An answer to this question was given only within the new renormalization group method. The simplest UV asymptotics of QED propagators obtained in our paper [3], for example, expression (1.7), agreed precisely with the results of Landau's group.

Within the renormalization group approach these results can be obtained in just a few lines of argument. To this end, the one-loop approximation

$$\bar{\alpha}^{(1)}_{PTh}(x,0;\alpha) = \alpha + \frac{\alpha^2}{3\pi}\ell + \dots \ , \qquad \ell = \ln x$$

of perturbation theory should be substituted into the right-hand side of the first equation in (1.6) to compute the generator $\beta(0,\alpha) = \psi(\alpha) = \alpha^2/3\pi$, followed by an elementary integration.

[6] Modern notation is used in this quotation from [2]

[7] These results were obtained under arbitrary covariant gauge.

Moreover, starting from the two-loop expression

$$\bar{\alpha}^{(2)}_{PTh}(x,0;\alpha)=\alpha+\frac{\alpha^2}{3\pi}\ell+\frac{\alpha^2}{\pi^2}\left(\frac{\ell^2}{9}+\frac{\ell}{4}\right)+\dots\,,$$

we arrive at the second renormalization group approximation (1.8) corresponding to the summation of the next-to-leading UV logarithms. This demonstrates that the RG method is a regular procedure, within which it is quite easy to estimate the range of applicability of the results.

The second–order renormalization group solution (1.8) for the effective coupling constant first obtained in [3] contains the nontrivial log–of–log dependence which is now widely known as the two–loop approximation for the running coupling constant in quantum chromodynamics (QCD).

Before long [4] this approach was formulated for the case of QFT with two coupling constants g and h, namely, for a model of pion–nucleon interactions with self-interaction of pions [8]. The following system of two coupled equation:

$$\bar{g}^2\left(x,y;g^2,h\right)=\bar{g}^2\left(\frac{x}{t},\frac{y}{t},\bar{g}^2(t,y;g^2,h),\bar{h}\left(t,y;g^2,h\right)\right)\ ,$$
$$\bar{h}\left(x,y;g^2,h\right)=\bar{h}\left(\frac{x}{t},\frac{y}{t},\bar{g}^2\left(t,y;g^2,h\right),\bar{h}\left(t,y;g^2,h\right)\right)\ .$$

was first obtained. The corresponding system of nonlinear differential equations from [4] was used in [76] to carry out the UV analysis of the renormalizable model of pion-nucleon interactions based on one-loop perturbative computations.

In [2, 3] and [4] the renormalization group approach was thus directly connected with practical computations of the UV and IR asymptotics. Since then this technique, known as the *renormalization group method* (RGM), has become the sole means of asymptotic analysis in local QFT.

A1.4 *Other early RG applications*

The first general theoretical application of the RG method was made in the summer of 1955 in connection with the (then topical) so-called ghost pole problem (also known as the "zero-charge trouble"). This effect, first discovered in QED [77, 78], was at first thought [78] to indicate a possible difficulty in quantum electrodynamics, and then [75, 79] as a proof of the inconsistency of the whole local QFT.

However, the renormalization group analysis of the problem carried out in [80] on the basis of (1.2) demonstrated that no conclusion obtained with the aid of finite–order computations within perturbation theory can be regarded as a complete proof. This corresponds precisely to the impression, one can get when comparing (1.7) and (1.8).

[8] It is essential that for the Yukawa PS-PS πN–interaction $\sim g$ to be renormalizable, it is necessary to add to the Lagrangian a quartic pion self-interaction term with an independent, that is, a second, coupling constant h. At that time this was not fully recognized: compare the given system with equations $(4.19)'$-$(4.21)'$ in [6], and the discussion in [75].

In the mid 50s this result was very significant, for it restored the reputation of local QFT. Nevertheless, in the course of the following decade the applicability of QFT in elementary particle physics remained doubtful in the eyes of many theoreticians.

In the general case of arbitrary covariant gauge the renormalization group analysis in QED was carried out in [81]. Here the problem is that the charge renormalization is connected only with the transverse part of the photon propagator. Therefore, under nontransverse (for example, Feynman) gauge the Dyson transformation has a more complex form. Logunov proposed to solve this problem by considering the gauge parameter is another coupling constant.

Ovsyannikov [17] found the general solution of the functional RG equations taking mass into account:

$$\Phi(y, \alpha) = \Phi\left(y/x, \bar{\alpha}(x, y; \alpha)\right)$$

in terms of an arbitrary function Φ of two arguments, reversible in its second argument. To solve the equations, he used the differential group equations represented as linear partial differential equations of the form[9]:

$$\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y, \alpha)\frac{\partial}{\partial \alpha}\right\}\bar{\alpha}(x, y, \alpha) = 0 \ .$$

The results of this "period of pioneers" were collected in the chapter "Renormalization group" in the monograph [8], the first edition of which appeared in 1957 [10], and very quickly acquired the status of "quantum–field folklore".

A2 Further RG development

A2.1 *Quantum field theory*

The next decade brought a calm period, during which there was practically no substantial progress in the renormalization group method. An important exception, which ought to be mentioned here, was Weinberg's article [83], in which the idea of a *running mass* of a fermion was proposed. If considered from the viewpoint of [81], this idea can be formulated as follows:

> *any parameter of the Lagrangian can be treated as a (generalized) coupling constant, and so can be included into the renormalization group formalism.*

However, the results obtained in the framework of this approach turned out the same as before. For example, the most familiar expression for the fermion running mass

$$\bar{m}(x, \alpha) = m_\mu \left(\frac{\alpha}{\bar{\alpha}(x, \alpha)}\right)^\nu \ ,$$

[9]Which are now known as the Callan—Symanzik equations.

[10]Shortly after that is was translated into English and French [8].

in which the leading UV logarithms are summed, was known for the electron mass in QED (with $\nu = 9/4$) since the mid 50s (see [74] [3]).

New possibilities for applying the RG method were discovered when the technique of operator expansion at small distances (on the light cone) appeared. The idea of this approach stems from the fact that the RG transform, regarded as a Dyson transformation of the renormalized vertex function, involves the simultaneous scaling of all its invariant arguments (normally, the squares of the momenta) of this function. The expansion on the light cone, so to say, "separates the arguments", as a result of which it becomes possible to study the physical UV asymptotic behaviour by means of the expansion coefficients (when some momenta are fixed and lie on mass shell). The argument–separation method for functions of several variables proposed by Wilson makes it possible to study a number of cases important from the physical point of view.

The end of the calm period can be marked well enough by the year 1971, when the renormalization group method was applied in the quantum theory of non-Abelian gauge fields, in which the famous effect of asymptotic freedom was discovered [82].

The renormalization group expression

$$\bar{\alpha}_s^{(1)} = \frac{\alpha_s}{1 + \beta_1 \ln x} \, ,$$

for the effective coupling constants $\bar{\alpha}_s$ in QCD, computed in the one-loop approximation, exhibits a remarkable UV asymptotic behaviour thanks to β_1 being positive. This expression implies, in contrast to Eq. (1.7), that the effective QCD constant decreases as x increases and tends to zero in the UV limit. This discovery, which has become technically possible only because of the RG method use, is the most important physical result obtained with the aid of the RG approach in particle physics.

Appendix B

Appendix II

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